One Repeat Point Gives a Closed, Unbounded Ultrafilter on ω_1

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Abstract

We prove that the following are equiconsistent:

1. ZF + DC + the closed unbounded filter on ω_1 is an ultrafilter.
2. ZFC + there is a cardinal κ with a weak repeat point.

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1 Introduction

Solovay (see [2, theorem 28.2]) proved in the late 1960s that the Axiom of Determinacy implies that ω_1 is measurable, and that the measure on ω_1 is generated by the filter $\mathcal{F}_{\mathrm{NS}}$ of closed unbounded subsets of ω_1. Given this fact, it was natural to try to prove the same conclusion directly from large cardinal properties. This was accomplished in [3]: Say that a measure W on κ is a *weak repeat point* if for every set $B \in W$ there is a measure $U \in \mathrm{ult}(V, W)$ such that $B \in U$. The main result of [3] constructs a model in which the closed unbounded filter $\mathcal{F}_{\mathrm{NS}}$ on ω_1 is an ultrafilter, assuming something stronger than a weak repeat point which is a limit of κ^+ many weak repeat points. On the other hand it is shown in [5, theorem 0.2] that if $\mathcal{F}_{\mathrm{NS}}$ is an ultrafilter then $K \cap \mathcal{F}_{\mathrm{NS}}$ is a weak repeat point in the core model K. A large gap remained between these two results, and in this paper we resolve this gap by showing that the lower bound is correct:

Theorem 1.1. *If ZFC + "W is a weak repeat point on κ" is consistent, then so is ZF + DC + "$\mathcal{F}_{NS}$ is an ultrafilter."*

The basic outline of the proof of theorem 1.1 is the same as that of the main result of [3]. The repeat point W on κ in V is used to guide the construction so that the closed unbounded filter $\mathcal{F}_{\mathrm{NS}}$ of the final model contains W. The main part of the construction uses iterated forcing to add a sequence $\vec{C} = \langle C_\gamma : \gamma < \kappa^+ \rangle$ of closed, unbounded subsets of κ such that every set $B \in W$ almost contains some member of $\vec{C}$. This is followed by a Levy collapse to obtain a model M in which $\kappa = \omega_1^M$, and the final model N is obtained as a submodel of M. The model N is defined so that each set C_γ in $\vec{C}$ is a member of N and $\omega_1^N = \kappa$, but the axiom of choice fails in N. The final part of the proof uses the homogeneity of the forcing to show that every subset of κ in N is either almost contained in or almost disjoint from one of the sets C_γ; thus N satisfies that $\mathcal{F}_{\mathrm{NS}}^N$ is a ultrafilter extending W.

The main difference between [3] and this paper is in the forcing used to construct the sequence $\vec{C}$. In [3] we used a slight modification of Radin's forcing from [9]. If W is a weak repeat point on κ and $B \in W$ then Radin's forcing, as applied to measurable cardinals (see [4]), can be used to add a closed unbounded subset $C \subset B$ while preserving the measure W in the sense that there is a measure $W' \supset W$ in $V[C]$. The proof of [3] iterated this forcing κ^+ times to generate the sequence $\vec{C}$; and since each iteration destroys the measures up to a repeat point this iteration requires κ^+ repeat points below the final repeat point W which guides the construction.

The inspiration for the current paper came from work of Gitik, who showed in [1] that $o(\kappa) \geq \kappa$, an assumption much weaker than a weak repeat point, is sufficient to construct a forcing which adds a closed unbounded subset of κ while preserving all measures on κ which have order at least κ. This raised the possibility of adapting the proof of [3] by iterating Gitik's forcing, instead of that of Radin, in order to obtain the closed unbounded sets $\vec{C}$. One difficulty

with this plan is that Gitik's forcing, unlike that of Radin, requires as a preliminary forcing the use of a modified backward Easton forcing to add new closed unbounded subsets of cardinals below κ. Because of this backward Easton forcing, Gitik's forcing does not seem to have the homogeneity necessary for the last step of the proof; however in [7] we present a modification of Gitik's forcing which resembles more closely a ordinary Easton support product forcing: the modified forcing has the property that if $\gamma < \kappa$ then the preliminary forcing $\mathcal{R}_\kappa$ can be regarded as a product $\mathcal{R}_{\gamma+1} \times \mathcal{R}_{\gamma+1,\kappa}$, in which the first factor $\mathcal{R}_{\gamma+1}$ adds subsets of cardinals $\lambda \leq \gamma$ while the second factor $\mathcal{R}_{\gamma+1,\kappa}$ adds subsets of cardinals λ in the interval $\gamma < \lambda < \kappa$ but does not add new subsets of γ. This modification gives the necessary homogeneity.

This paper assumes that the reader is familiar with [7]. In Section 2 we recall the definition and basic properties of the iterated forcing introduced in [7], with one definition slightly modified to deal with the fact that we are here adding many new closed unbounded subsets, instead of one, for each cardinal λ. Section 3.2 deals with definition of the forcing Q_λ used at each Mahlo cardinal $\lambda < \kappa$ to form the iteration. Most of the material in this section is new to this paper; however the definitions are heavily motivated by those of [7] and for the climax of the proofs of two crucial lemmas, 3.62 and 3.67, we will refer the reader back to the proofs of analogous results in [7].

The final model is presented in Section 4. The first part of this section defines the forcing used to define the closed unbounded subsets of κ. This forcing is essentially a direct limit Q^*_κ of forcings $\langle\, Q_\lambda^{\mathcal{E}_\kappa \restriction \eta} : \eta < \kappa^+ \,\rangle$ which are defined just like the forcing Q_λ used at cardinals $\lambda < \kappa$. This direct limit is guided by the sequence $\mathcal{E}_\kappa$ which is, in part, an enumeration $\langle\, E_\eta : \eta < \kappa^+ \,\rangle$ of the repeat point W. The final, longer part of section 4 uses ordinal definability to construct the desired submodel N of the generic extension so that N satisfies that the closed unbounded filter is an ultrafilter.

Notation. Our notation is generally standard. In forcing, we use $p \leq p'$ to mean that p is stronger than p', and we use $P \equiv P'$ to indicate that the forcing notions P and P' are equivalent.

We frequently use the nonstandard notation $f{\downharpoonleft}\nu$ as a complement to the usual restriction $f\restriction\nu$; using $f{\downharpoonleft}\nu$ for $f\restriction\{\,\xi \in \operatorname{domain}(f) : \xi \geq \nu\,\}$. If X is a set of sequences then we write $X\restriction\gamma$ for $\{\, q\restriction\gamma : q \in X \,\}$.

1.1 Canonical coherence

The forcing will depend heavily on the fact that ordinals up to α^+ have canonical representatives in ultrapowers by measures on α. For this it will be convenient to use the machinery developed in section 3.1 of [8], which relies on $\square_\kappa$, and for this purpose we will assume that $V = L[\mathcal{W}]$, where $\mathcal{W}$ is a sequence of measures having a repeat point on κ. Then $\square_\lambda$ holds for every cardinal λ, and furthermore the square sequences are preserved by embeddings: there are sequences $C^\kappa = \langle\, C^\kappa_\alpha : \kappa < \alpha \leq \kappa^+ \,\rangle$ witnessing the truth of $\square_\kappa$, such that

whenever $X \prec_{\Sigma_1} H_{\lambda^+}$, $\lambda' = X \cap \lambda$ is a cardinal less than λ, and $\pi\colon X \cong X'$ is the collapse map then $\pi(C^{\lambda}_{\alpha}) = C^{\lambda'}_{\pi(\alpha)}$ for all $\alpha > \lambda$ in X.

Lemma 1.2. *There is a sequence of sets $A_{\alpha,\xi}$, defined for any cardinal λ and ordinals $\xi < \lambda \leq \alpha < \lambda^+$, such that* (i) *$\langle A_{\alpha,\xi} : \xi < \lambda \rangle$ is an increasing sequence of subsets of α,* (ii) *$|A_{\alpha,\xi}| \leq |\xi|$,* (iii) *$A_{\alpha,\xi} = \bigcup_{\xi'<\xi} A_{\alpha,\xi'}$ if ξ is a limit ordinal,* (iv) *$\alpha = \bigcup_{\xi<\lambda} A_{\alpha,\xi}$.* (v) *$A_{\alpha',\xi} = A_{\alpha,\xi} \cap \alpha'$ whenever $\lambda \leq \alpha' \in A_{\alpha,\xi} \cup \lim(A_{\alpha,\xi})$. The sets $A_{\alpha,\xi}$ can be defined, uniformly in λ, α and ξ, from the sequences of sets witnessing $\square_\lambda$.*

Proof. The construction of the sets $A_{\alpha,\xi}$ is given in section 3.1 of [8]. The only difference in our usage here is that we will be using the sets $A_{\alpha,\xi}$ for all cardinals $\lambda \leq \kappa$, and in order to avoid having to complicate the notation by explicitly specifying λ we do not consider the case of $\alpha < \lambda$. $\square$

Lemma 1.3. *Suppose that $X \prec_{\Sigma_0} (H_{\lambda^+}, \vec{A})$, $\lambda' = X \cap \lambda$ is a cardinal less than λ, and the collapse map $\pi\colon X \cong M$ preserves the square sequence. Then* (i) *$A_{\pi(\alpha),\xi} = \pi(A_{\alpha,\xi})$ for all $\alpha \in X \setminus \lambda$ and $\xi \in X \cap \lambda$, and* (ii) *if $\gamma \in X \setminus \lambda$ then $X \cap \gamma = A_{\gamma,\lambda}$.* $\square$

Definition 1.4. 1. We write $\alpha' \propto_\nu \alpha$ if $\alpha' = \alpha$ or $\alpha' \in A_{\alpha,\nu}$.

I just added "$\alpha' = \alpha$". I'll need to check, as I go through this, if this requires changes elsewhere. It would be nice if the symbol allowed $<$ and $\leq$ variants.

Definition 1.5. If $z \in \mathrm{Hered}_{\lambda^+}$ then we write $\bar{z} = z{\downarrow}\nu$ if $\bar{z} = \pi(z)$, where $\pi\colon X \cong M$ is any transitive collapse map as in Lemma 1.3.

Proposition 1.6. *1. The object $z{\downarrow}\nu$ does not depend on the choice of the set X.*

2. $(x{\downarrow}\nu){\downarrow}\nu' = x{\downarrow}\nu'$ whenever the left side is defined.

3. If $z \subset \lambda$ then $z{\downarrow}\nu = z \cap \nu$.

4. If U is a measure on λ and $i\colon V \to M = \mathrm{ult}(V,U)$ then $M \models z = i(z){\downarrow}\lambda$ for all $z \in \mathrm{Hered}_{\lambda^+}$. $\square$

Proposition 1.7. *1. The relation $\propto_\nu$ is transitive for any $\nu < \lambda$.*

2. If $\nu < \nu' < \lambda$ and $\alpha \propto_\nu \alpha$ then $\alpha \propto_{\nu'} \alpha'$.

3. If $\alpha \propto_\nu \alpha'$ and $\alpha'{\downarrow}\nu$ is defined then $\alpha{\downarrow}\nu$ is also defined. $\square$

Definition 1.8. If $y_\gamma \subset \lambda$ for $\gamma < \beta$ then we write $\Delta_{\gamma<\beta}\, y_\gamma$ for the diagonal intersection

$$\mathop{\Delta}_{\gamma<\beta} y_\gamma = \left\{ \nu < \lambda : \nu \in \bigcap \{\, y_\gamma : \gamma < \beta \text{ and } \gamma \propto_\nu \beta \,\} \right\}.$$

If σ is the $\lessdot$-least 1-1 map from λ onto β, then equivalently

$$\mathop{\Delta}_{\gamma<\beta} y_\gamma = \left\{ \nu < \lambda : \nu \in \bigcap_{\gamma \in \sigma``\nu} y_\gamma \right\}.$$

Note: The trees won't be "almost continuously decreasing". In fact, if $\{\, \iota : \iota \prec_\nu \gamma \,\} = \{\, \iota : \iota \prec_\nu \gamma' \,\}$ then at most one of $\nu \in C_{\lambda,\gamma}$ and $\nu \in C_{\lambda,\gamma'}$ hold. (why?)

Definition 1.9. A sequence $\vec{E} = \langle\, E_\gamma : \gamma < \beta \,\rangle$ of subsets of α, for $\beta \leq \alpha^+$ is *decreasing* if $E_{\alpha'} \setminus \nu \subset E_\alpha$ whenever $\alpha' \propto_\nu \alpha$, and it is *continuously decreasing* if $E_\alpha = \Delta_{\nu<\alpha} E_\nu$ for limit ordinals α.

1.2 Layered tree sequences

Layered tree sequences correspond to the tree sequences of [7]. The primary difference between the two is that where the latter specifies a single closed subset of each cardinal λ in its domain, the former specifies a sequence $\langle\, C_{\lambda,\iota} : \iota < \lambda^+ \,\rangle$ of such sets. A secondary difference is that the layered tree structures are not quite trees. If they followed the tree property of [7] then they would satisfy

$$C_{\lambda,\iota} \cap \gamma = C_{\gamma,\iota\downarrow\gamma} \text{ for each } \gamma \in C_{\lambda,\iota}. \tag{1}$$

This would conflict with the homogeneity arguments in the final section of this paper, where we need to be able to make arbitrary changes to initial segments of $\vec{C}$ and still have a generic sequence. Hence the equation (1) will only hold for sufficiently large $\gamma \in C_{\lambda,\iota}$.

The sequence $\vec{C}$ will be constructed by an iterated forcing, in which the sequence $\vec{C}_\lambda = \langle\, C_{\lambda,\iota} : \iota < \lambda^+ \,\rangle$ comes from a Prikry type forcing over $V[\vec{C}{\restriction}\lambda]$. In this section we will define a partial order $(P^{\vec{C}}_\lambda, \prec)$, depending on $\vec{C}{\restriction}\lambda$. A condition $\vec{\nu}$ in this forcing will serve the same function in the forcing of this paper as the finite set a does in a condition (a, A) of the Prikry forcing. The forcing $P^{\vec{C}}_\lambda$ will be used in the definition 1.29 of a layered tree sequence. The universe of the partial order $(P^{\vec{C}}_\lambda)$ is a set P_λ which does not depend on $\vec{C}$:

Definition 1.10. The members of P_λ are sequences $\vec{\nu} = \langle\, \nu_\iota : \iota < \lambda^+ \,\rangle$ of cardinals less than λ such that for some finite set $I = I^{\vec{\nu}}$, which we call the *support* of $\vec{\nu}$, (i) $\{\iota, (\vec{\nu}{\restriction}(I \setminus \iota + 1))\}{\downarrow}\nu_\iota$ is defined for each $\iota \in I$, (ii) $\langle\, \nu_\iota : \iota \in I \,\rangle$ is strictly decreasing, and (iii) $\nu_\iota = \max\{\, \nu_{\iota'} : \iota' \in I \setminus \iota \text{ and } \iota \propto_{\nu_\iota} \iota' \,\}$ for all $\iota \in \lambda^+ \setminus I$, (with $\nu_\iota = 0$ if this set is empty).

The sequence $\vec{\nu}$ as a whole will not, in general, be decreasing; however since $0 \propto_\nu \alpha$ for all α and all $\nu > 0$ it is alway true that $\nu_0 = \max \operatorname{range}(\vec{\nu})$.

Actually, I think it *is* nonincreasing except where $\nu_\gamma = 0$. If $\gamma < \gamma'$ are in $I^{\vec{\nu}}$ then $\gamma'{\downarrow}\nu_\gamma$ exists, which implies $\gamma \propto_{\nu_\gamma} \gamma'$. Then for any $\gamma'' < \gamma$ such that $\gamma'' \propto_{\nu_{\gamma'}} \gamma'$ we also have $\gamma' \propto_{\nu_\gamma} \gamma$ so $\nu_{\gamma''} \geq \nu_\gamma$.

7/23/07 — The forcing $P^{\vec{C}}_\lambda$ will have the property that if $\vec{\nu}' \prec \vec{\nu}$ and $\nu'_\gamma > \nu_\gamma$ then $\nu'_{\gamma'} \geq \nu'_{\gamma''}$ whenever $\gamma' \leq \gamma < \gamma''$ and $\nu'_{\gamma'} > 0$. However maximal elements of $P^{\vec{C}}_\lambda$ need not be nondecreasing on nonzero members.

Although the members $\vec{\nu}$ of P_λ nominally have length λ^+, we may regard them as members of $\mathrm{Hered}_{\lambda^+}$ since $\nu_\gamma = 0$ for all $\gamma > \sup I^{\vec{\nu}}$. By this convention we can identify the finite sequence $\vec{\nu}{\restriction}(I \setminus \iota+1)$ with $\vec{\nu}{\lfloor}\iota+1$ and restate clause (i) as "$(\vec{\nu}{\lfloor}\iota+1){\downarrow}\nu_\iota$ is defined".

Proposition 1.11. *Any sequence $\vec{\nu} \in P_\lambda$ has a unique support.*

Proof. $I^{\vec{\nu}} = \{\, \iota < \lambda^+ : \nu_\iota > 0 \;\;\&\;\; \forall \iota' > \iota \; \nu_{\iota'} < \nu_\iota \,\}$. □

Proposition 1.12. *Suppose that $\vec{\nu}$ and $\vec{\nu}'$ are in P_λ, and that $\gamma < \lambda^+$ is such that $\nu_\gamma > \nu'_{\gamma'}$ for all $\gamma' > \gamma$ and $\gamma \propto_{\nu_\gamma} \gamma'$ for all $\gamma' \in I^{\vec{\nu}'} \setminus \gamma+1$. Then $(\vec{\nu}{\restriction}\gamma+1)^\frown(\vec{\nu}'{\lfloor}\gamma+1)$ is a member of P_λ with support $(I^{\vec{\nu}} \cap \gamma) \cup \{\gamma\} \cup (I^{\vec{\nu}' \setminus \gamma+1})$.* □

6/26/07 — I think this may come in when I look at genericity of initial segments of $\vec{C}_\lambda$. Fix $\vec{\nu} \in F_\lambda$ and look at the set F'_λ of sequences $(\vec{\nu}'{\restriction}\gamma+1)^\frown(\vec{\nu}{\lfloor}\gamma+1)$ where $\vec{\nu}' \in F_\lambda$ and ν'_γ satisfies the conditions of the proposition. Then F'_λ should be (or at least generate) a filter such that the sequence $(\vec{C}_\lambda{\restriction}\gamma+1)^\frown\langle\, C_{\lambda,\iota} \cap \nu_\iota : \iota > \gamma \,\rangle$ is equal to $\mathrm{ext}^{\vec{C}}_\lambda(F'_\lambda)$.

Note that, given $\vec{\nu}'$ and γ, there is $\nu < \lambda$ such that the hypothesis of proposition 1.12 holds for all $\vec{\nu}$ with $\nu_\gamma \geq \nu$. This proposition becomes false if $\gamma+1$ is replaced with a limit ordinal.

The following definition gives a preliminary characterization of the sequence $\vec{C}$:

Definition 1.13. A *layered sequence of closed sets* is a sequence $\vec{C}$ with domain $\{\, (\lambda, \iota) : \alpha < \lambda < \xi \;\;\&\;\; |\lambda| = \lambda \;\;\&\;\; \iota < \lambda^+ \,\}$ for some interval (α, ξ) of cardinals, such that for each cardinal in (α, ξ) the sequence $\vec{C}_\lambda = \{\, C_{\lambda,\iota} : \iota < \lambda^+ \,\}$ is a decreasing sequence of closed subsets of λ.

We will frequently regard as $\vec{C}$ as a function with domain the set of cardinal in (α, ξ). The notation $\vec{C}_\lambda$ above is an example of this. In addition, we will write $\vec{C}{\restriction}\lambda$ for $\vec{C}{\restriction}\Big\{\, (\nu, \gamma) \in \mathrm{domain}(\vec{C}) : \nu < \lambda \,\Big\}$.

The sets need not be *continuously* decreasing (this is immediate from the fact that $C_{\gamma,\iota} = \emptyset$ for all but boundedly many $\iota < \gamma^+$).

As of May 06 I thought that the following is true at least for all sufficiently large ν: If $\nu \in C_{\gamma,\iota}$ for some ι then there is ι_0 such that $\nu \in C_{\gamma,\iota}$ iff $\iota \propto_\nu \iota_0$.

6/18/07 — Of course to the extent that can have $\vec{C}$ diagonally decreasing, we can have it continuously diagonally decreasing. Define $\vec{C}'$ by $C'_{\lambda,\beta} = \Delta_{\beta'<\beta} C_{\lambda,\beta'}$ if β is a limit ordinal, and $C'_{\lambda,\beta+1} = C_{\lambda,\beta}$. If $\vec{C}$ is diagonally decreasing, then $\vec{C}'$ is continously diagonally decreasing.

However this would complicate things without any real benefit.

The next two definitions further delineate the way in which the new sequence $\vec{C}_\lambda$ at λ should cohere with the previously constructed sequence $\vec{C}\restriction\lambda$.

In the following definitions, λ is a cardinal and $\vec{C}$ is a layered sequence of closed sets. For notational convenience we allow $\lambda \in \operatorname{domain}(\vec{C})$; however the definitions only depend on $\vec{C}\restriction\lambda$.

Definition 1.14. Suppose that $\vec{C}$ is a layered sequence of closed sets and $\vec{\nu} \in P_\lambda$. We define the notions of *joined* and *smoothly joined* by recursion on λ:

1. If $C \subset \lambda$ and $\gamma < \lambda^+$ then we say C is *γ-joined to $\vec{C}$ at $\vec{\nu}$* if
   (a) If $\nu_\gamma = 0$ then $\min(C) > \nu_0$.
   (b) If $\nu_\gamma > 0$ and $\nu_\gamma \notin \operatorname{domain}\vec{C}$ then $C \cap (\nu_\gamma + 1) = \{\nu_\gamma\}$.
   (c) If $\nu_\gamma > 0$ and $\nu_\gamma \in \operatorname{domain}\vec{C}$ then for each $\eta \in I^{\vec{\nu}}$,
   $$C \cap (\nu', \nu_\eta] = \begin{cases} \{\nu_\eta\} \cup C_{\nu_\eta,\gamma\downarrow\nu_\eta} \setminus (\nu'+1) & \text{if } \gamma \propto_{\nu_\eta} \eta \\ \varnothing & \text{otherwise,} \end{cases}$$
   where $\nu' = \max(\operatorname{range}(\vec{\nu}) \cap \nu_\eta)$. In particular, if $\nu_\gamma < \nu_0$ then $C \cap \nu_0 + 1 \subseteq \nu_\gamma + 1$.
2. We say that C is *smoothly γ-joined to $\vec{C}$ at $\vec{\nu}$* if C is γ-joined to $\vec{C}$, and, in addition, $C \cap \nu = C_{\nu,\gamma\downarrow\nu}$ and $\vec{\nu}\downarrow\nu$ exists for each ordinal $\nu \in C \setminus (\nu_\gamma + 1)$.

   6/24/07 — I originally added this:

   "$\vec{C}_\nu$ is smoothly joined to $\vec{C}$ at $\vec{\nu}\downarrow\nu$."

   However I don't remember my reason for thinking this is needed, and can't think of anywhere this is used. In addition, it seems to make lemma 1.17 false — at least I can't see how to fix the proof. The problem is that I don't see why $\vec{C}_\lambda$ should be smoothly joined to $\vec{\nu}'$, even if $\nu'_{\gamma'} \geq \mu_{\gamma'}$ for all γ', because I don't see why this added condition should hold for $\nu \in C_{\lambda,\gamma} \cap (\nu'_\gamma, \nu_\gamma]$.

   Note that you automatically get the same statement for $\vec{C}_\nu\restriction(\gamma\downarrow\nu)$.

   6/29/07 — This seems to have been added for the proof of lemma 4.18. However, it doesn't seem to be needed.
3. A sequence $\vec{D} = \langle D_\gamma : \gamma < \lambda^+ \rangle$ of subsets of λ is (smoothly) joined to $\vec{C}$ at $\vec{\nu}$ if D_γ is (smoothly) γ-joined to $\vec{C}$ at $\vec{\nu}$ for each $\gamma < \lambda^+$.

[6/24/02] It's not clear to me that I ever use unsmooth joins.

We are now prepared to define a partial ordering $(P^{\vec{C}}_\lambda, \prec)$ so that a suitable filter $\mathcal{F}$ in $(P^{\vec{C}}_\lambda, \prec)$ will yield a sequence $\vec{C}_\lambda$ which is smoothly attached to $\vec{C}$ at any $\vec{\nu} \in \mathcal{F}$. Such a filter $\mathcal{F}$ will have a maximal member (that is, a sequence $\vec{\nu}$ which is minimal in the lexicographic order on sequences of ordinals). This maximal member need not be the constant sequence $\vec{0}$, and thus $\vec{C}_\lambda$ may not be smoothly joined to $\vec{C}$ at $\vec{0}$.

Definition 1.15. Suppose that $\vec{C}$ is layered sequence of closed sets, and λ is a cardinal. We define a partial order $P^{\vec{C}}_\lambda = (P_\lambda, \prec)$ as follows:

1. Assume that $\vec{\nu} \in P_\lambda$, and $\nu > \max \vec{\nu}$. Then $\mathrm{add}(\vec{\nu}, \eta, \nu)$ is the sequence $\vec{\nu}' \in P_\lambda$ with support $I^{\vec{\nu}'} = \{\eta\} \cup I^{\vec{\nu}} \setminus \eta$ which is defined by $\nu'_\eta = \nu$ and $\vec{\nu}' \lfloor \eta + 1 = \vec{\nu} \lfloor \eta + 1$.

2. The ordering $\prec$ of $P^{\vec{C}}_\lambda$ is the least transitive relation such that $\mathrm{add}(\vec{\nu}, \eta, \nu) \prec \vec{\nu}$ whenever

$$\{\vec{\nu}, \eta\}{\downarrow}\nu \text{ is defined} \quad \& \quad \vec{C}_\nu \text{ is smoothly joined to } \vec{C} \text{ at } \vec{\nu}{\downarrow}\nu \\ \& \quad \forall \gamma > \eta{\downarrow}\nu \; C_{\nu,\gamma} \subset (\vec{\nu}{\downarrow}\nu)_\gamma + 1. \tag{2}$$

7/2007 — Why do I need to add that $C_{\nu,\gamma} \subset \nu_\gamma + 1$ for $\gamma > \eta$? Because this is needed to get the "generic" $\vec{C}_\lambda$ to be smoothly joined at $\mathrm{add}(\vec{\nu}, \eta, \nu)$. (Actually I use it all over.)

Proposition 1.16. *If $\vec{\nu}' \prec \vec{\nu}$ then $\vec{\nu}' = \vec{\mu}(\eta, \nu)^\frown(\vec{\nu} \lfloor \eta + 1)$ where $\vec{\mu}(\eta, \nu)$ is the sequence $\vec{\mu}$ with $I^{\vec{\mu}} = \{\eta\}$ and $\mu_\eta = \nu$.*

Proof. This follows from proposition 1.12 and the definition of $(P^{\vec{C}}_\lambda, \prec)$. $\square$

7/2007 — If $\vec{\mu} \prec \vec{\nu}$ then $\vec{C}_{\sup(\mathrm{range}\, \vec{\mu})}$ is smoothly joined to $\vec{C}$ at $\vec{\nu}$.

Lemma 1.17. *If $\vec{C}_\lambda$ is smoothly joined to $\vec{C}$ at both $\vec{\nu}$ and $\vec{\mu}$, and $\nu_\gamma > \mu_\gamma$ for some $\gamma < \lambda^+$, then $\vec{\nu} \prec \vec{\mu}$.*

Proof. Note that if γ is the largest member of $I^{\vec{\nu}}$ such that $\nu_\gamma > \mu_\gamma$ then definition 1.14(1c) implies that $\nu_{\gamma'} > \mu_0$ for all $\gamma' \in I^{\vec{\nu}} \cap (\gamma + 1)$. Since the same is true with $\vec{\nu}$ and $\vec{\mu}$ interchanged, it follows that $\nu_\gamma \geq \mu_\gamma$ for all $\gamma < \lambda^+$.

Now we prove the lemma by induction on the number n of ordinals $\nu \in I^{\vec{\nu}}$ such that $\nu_\gamma > \mu_\gamma$. Let $\gamma = \min(I_{\vec{\nu}})$, and let $\vec{\nu}'$ be the sequence obtained by dropping this entry; that is, $I^{\vec{\nu}'} = I^{\vec{\nu}} \setminus \{\gamma\}$ and $\nu'_{\gamma'} = \nu_{\gamma'}$ for all $\gamma' > \gamma$. We consider two cases.

For the first case, we suppose that $\mu_{\gamma'} > \nu'_{\gamma'}$ for some $\gamma' < \lambda^+$, and show that this implies that $\vec{\nu} = \mathrm{add}(\vec{\mu}, \gamma, \nu_\gamma) \prec \vec{\mu}$. Since $\vec{\nu}' \lfloor \gamma + 1 = \vec{\nu} \lfloor \gamma + 1$, $\nu'_{\gamma'} < \mu_{\gamma'}$ implies $\gamma' \leq \gamma$. I claim that $\mu_{\gamma'} > \nu'_{\gamma'}$ implies that $\mu_{\gamma'} > \nu'_0$. Otherwise let γ'' be the largest member of $I^{\vec{\nu}'}$ such that $\nu_{\gamma''} \geq \mu_{\gamma'}$. Then $\gamma' \not\lessdot_{\nu_{\gamma''}} \gamma''$, since otherwise we would have $\nu'_{\gamma'} \geq \nu'_{\gamma''} = \nu_{\gamma''}$. Since $\vec{C}_\lambda$ is smoothly joined to $\vec{C}$ at

$\vec{\nu}$ it follows that $C_{\gamma'} \cap (\nu_{\gamma''}, \nu] = \emptyset$, where $\nu = \max(\{0\} \cup (\operatorname{range}(\vec{\nu}) \cap \nu))$, and this contradicts the fact that $\mu_{\gamma'} \in C_{\lambda,\gamma'}$.

It follows that $\mu_{\gamma''} = \nu_{\gamma''}$ for all $\gamma'' > \gamma$, as by the first paragraph $\nu_{\gamma''} > \mu_{\gamma''}$ would imply $\nu_{\gamma''} > \mu_0 \geq \mu_{\gamma'}$. Hence $\vec{\nu} = \operatorname{add}(\vec{\mu}, \gamma, \nu_\gamma)$. It remains to verify that $\operatorname{add}(\vec{\mu}, \gamma, \nu_\gamma) \prec \vec{\mu}$, and for this we need to check that $(\vec{\mu}{\restriction}\gamma + 1){\downarrow}\nu_\gamma$ exists. An argument like in the last paragraph confirms that this is true.

For the second case we assume that $\nu'_{\gamma'} \geq \nu_{\gamma'}$ for all γ'. In this case it is straightforward to verify that $\vec{\nu} = \operatorname{add}(\vec{\nu}', \gamma, \nu_\gamma) \prec \vec{\nu}'$ and that $\vec{C}_\lambda$ is smoothly joined to $\vec{C}$ at $\vec{\nu}'$. It follows from the induction hypothesis that $\vec{\nu}' \preceq \vec{\mu}$, and hence $\vec{\nu} \prec \vec{\mu}$.

See the note on definition 1.14 (if it's still there). This is the point were the added condition causes trouble.

□

Corollary 1.18. *If $\vec{C}$ is a layered sequence of closed sets and $\vec{D}$ is a sequence of closed subsets of λ then the set of $\vec{\nu} \in P_\lambda^{\vec{C}}$ such that $\vec{D}$ is smoothly connected to $\vec{C}$ at $\vec{\nu}$ is linearly ordered by $\prec$.* □

Corollary 1.19. *If $\vec{\nu} \in P^{\vec{C}}$ then $\{\, \vec{\mu} \in P_\lambda : \vec{\mu} \prec \vec{\nu} \,\}$ is linearly ordered by $\prec$.*

Proof. Suppose that $\vec{\nu} \prec \vec{\mu}_0$ and $\vec{\nu} \prec \vec{\mu}_1$ and that $\vec{\nu}$ is the $\prec$-largest such sequence. We will show that either $\vec{\nu} = \vec{\mu}_0$ or $\vec{\nu} = \vec{\mu}_1$.

If not, then there are sequences $\vec{\mu}'_0$ and $\vec{\mu}'_1$ such that the inequalities are witnessed by $\vec{\nu} = \operatorname{add}(\vec{\mu}'_i, \gamma, \nu_\gamma) \prec \vec{\mu}'_i \preceq \vec{\mu}_i$ for $i = 0, 1$, where $\gamma = \min(I^{\vec{\nu}})$. Then $\vec{C}_{\nu_\gamma}$ is smoothly joined to $\vec{C}$ at both $\vec{\mu}_0$ and $\vec{\mu}_1$, and it follows by lemma 1.17 that either $\vec{\mu}'_0 \preceq \vec{\mu}'_1$ or $\vec{\mu}'_1 \preceq \vec{\mu}'_0$. Either of these possibilities contradicts the choice of $\vec{\nu}$. □

Corollary 1.19 says that $(P_\lambda^{\vec{C}}, \prec)$ is a tree ordering, though one with multiple roots. Thus a filter in $(P_\lambda^{\vec{C}}, \prec)$ is a branch of this tree. The ordering $(P_\lambda^{\vec{C}}, \prec)$ is inversely well founded, since $\vec{\mu} \prec \vec{\nu}$ implies that $\max \vec{\mu} > \max \vec{\nu}$. It follows that every filter of $(P_\lambda^{\vec{C}}, \prec)$ has one of the roots of the tree as its maximal element.

Definition 1.20. Suppose that $\vec{C}$ is a layered sequence of closed sets and F is a filter in $(P_\lambda^{\vec{C}}, \prec)$. Then $\operatorname{ext}_\lambda^{\vec{C}}(F)$ is the sequence $\langle\, C_{\lambda,\iota} : \iota < \lambda^+ \,\rangle$ defined by setting $C_{\lambda,\iota}$ equal to the union of the sets $C_{\nu_\gamma, \gamma\downarrow\nu_\gamma} \setminus \nu + 1$ where $\vec{\nu} \in F$ and $\nu = \max(\operatorname{range}(\vec{\nu}) \cap \nu)$.

6/29/07 — Note that 0 is always in range($\vec{\nu}$), so this covers all cases. This could be used to simplify earlier statements.

It is straightforward to verify the following observation:

Proposition 1.21. *If $\vec{C}_\lambda = \operatorname{ext}_\lambda^{\vec{C}\restriction\lambda}(F)$ for some filter F_λ in $P_\lambda^{\vec{C}\restriction\lambda}$ then $\vec{C}_\lambda$ is smoothly joined to $\vec{C}$ at each $\vec{\nu} \in F_\lambda$.* □

The filters F we will be using for this construction will not be generic filters in $(P^{\vec{C}}_{\lambda}, \prec)$; however they will be projections of a generic filter in a Prikry style forcing obtained by using members of $(P^{\vec{C}}_{\lambda}$ together with side conditions.

Definition 1.22. A *layered tree sequence* is a layered sequence $\vec{C}$ of closed sets such that for each $\lambda \in \operatorname{domain}(\vec{C})$ there is a filter F_λ in $P^{\vec{C}}_{\lambda}$ such that $\vec{C}_\lambda = \operatorname{ext}^{\vec{C}\restriction\lambda}_{\lambda}(F_\lambda)$ and either $C_{\lambda,0}$ is unbounded in λ or else F_λ has a $\prec$-minimal member.

The requirement that $\vec{C}$ be a layered sequence of closed sets is redundent, since any sequence $\vec{C}$ constructed recursively from filters F_λ as specified in Definition 1.22 will be a layered sequence of closed sets. This fact follows by an induction on λ, in which the next Proposition is the limit step and the following Lemma 1.24 is the successor step.

Proposition 1.23. *Suppose that $\vec{C}$ is a sequence of limit length λ such that $\vec{C}\restriction\lambda'$ is a layered tree sequence for all $\lambda' < \lambda$. Then $\vec{C}$ is a layered tree sequence.* $\square$

Lemma 1.24. *Suppose that $\vec{C}$ is a layered tree sequence of height λ and $\vec{D} = \operatorname{ext}^{\vec{C}}_{\lambda}(F)$ where F is a filter in $P^{\vec{C}}_{\lambda}$ such that either $C_{\lambda,0}$ is unbounded in λ or else F_λ has a $\prec$-minimal member. Then $\vec{D}$ is a descending sequence of closed sets, and hence $\vec{C}^\frown\langle\lambda, \vec{D}\rangle$ is a layered tree sequence of height $\lambda + 1$.*

The proof of Lemma 1.24 will follow Lemma 1.28. In all of the intervening results, $\vec{C}$ and $\vec{D}$ are as specified in the statement of Lemma 1.24.

Proposition 1.25. *The sequence $\vec{D}$ is a descending sequence of subsets of λ.*

Proof. We need to show that if $\gamma \propto_\nu \gamma'$ and $\nu \in D_{\gamma'}$ then $\nu \in D_\gamma$. The assumption that $\nu \in D_{\gamma'}$ implies that there is some $\vec{\nu} \in F$ and $\gamma'' \in I^{\vec{\nu}}$ such that $\gamma' \propto_{\nu''_\gamma} \gamma''$ and $\nu \in \{\nu_{\gamma''}\} \cup C_{\nu_{\gamma''},\gamma'\downarrow\nu_{\gamma''}} \cap (\nu_{\gamma''}, \nu')$, where $\nu' = \max(\operatorname{range}\vec{\nu}) \cap \nu_{\gamma''}$. Since $\gamma \propto_\nu \gamma'$ and $\nu_{\gamma''} \geq \nu$ it follows that $\gamma \propto_{\nu_{\gamma''}} \gamma''$. If $\nu = \nu_{\gamma''}$ then it follows immediately that $\nu \in C_{\lambda,\gamma}$, and otherwise $\nu \in C_{\lambda,\gamma}$ follows from the fact the the sequence $\vec{C}_{\nu_{\gamma''}}$ is descending. $\square$

Proposition 1.26. *For all but boundedly many $\gamma < \lambda^+$ the set D_γ is empty. Furthermore, if $\gamma < \lambda^+$ and D_γ is unbounded in λ then so is $D_{\gamma'}$ for all $\gamma' < \gamma$.*

Proof. Since F is linearly ordered it has cardinality at most λ. Thus $\bigcup_{\vec{\nu}\in F} I^{\vec{\nu}}$ is bounded in λ^+, so $D_\gamma = \emptyset$ for all $\gamma > \sup(\bigcup_{\vec{\nu}\in F_\lambda} I^{\vec{\nu}})$.

Now suppose that $\gamma' < \gamma$ and D_γ is unbounded in λ. Let $\xi < \lambda$ be large enough that $\gamma' \propto_\xi \gamma$. Then by lemma 1.25, $D_\gamma \setminus \xi \subseteq D_{\gamma'}$, and thus $D_{\gamma'}$ is also unbounded in λ. $\square$

We will write $o^*(\vec{D})$ for the least ordinal γ such that D_γ is bounded in λ. Thus D_γ is unbounded in λ if and only if $\gamma < o^*(\vec{D})$.

7/2007 — This is the current notation. I'd previously used $o^*(\lambda)$ for this. That is bad when used for $o^*(\vec{C}_\lambda)$ and even worse here.

Proposition 1.27. *$\bigcup_{\gamma \geq o^*(\vec{D})} D_\gamma$ is bounded in λ.*

Proof. Suppose that the proposition is false and fix $\vec{\nu} \in F$ and $\gamma \in I^{\vec{\nu}} \setminus o^*(\vec{D})$ such that $\bigcup_{\gamma' > \gamma} D'_\gamma$ is unbounded in λ. Then for every $\xi < \lambda$ there is $\vec{\nu}' \in F$ and $\gamma' \in I^{\vec{\nu}'} \setminus \gamma$ such that $\nu'_{\gamma'} > \xi$. However the definition of $(P^{\vec{C}}_\lambda, \prec)$ implies that in this case $\vec{\nu}{\downarrow}\nu'_{\gamma'}$ exists, which implies that $\gamma \propto_{\nu'_\gamma} \gamma'$ and hence $\nu_{\gamma'} \in D_\gamma$. Thus D_γ is unbounded in λ, contradicting the choice of γ. $\square$

Lemma 1.28. *There is $\vec{\nu} \in F$ such that $\nu_\gamma = \sup(D_\gamma)$ for all $\gamma \geq o^*(\vec{D})$. More generally, for any $\alpha < \lambda$ there is $\vec{\nu} \in F$ such that $\nu_\gamma = \sup(D_\gamma)$ whenever $\gamma \geq o^*(\vec{D})$ and $\nu_\gamma = \sup(D_\gamma) \cap \alpha$ whenever $\nu_\gamma < \alpha$.*

It follows that each of the sets D_γ is closed.

Proof. Since any sequence $\vec{\nu}$ satisfying the second sentence with $\alpha \geq \sup \bigcup_{\gamma \geq o^*(\vec{D})} D_\gamma$ will satisfy the first sentence, it will be sufficient to prove the second sentence.

If D_0 is bounded in λ, then the required minimal member of F is as required, so we can assume that D_0 is unbounded in λ. Thus there is $\vec{\nu} \in F$ with $\nu_0 > \alpha$. I claim that $\vec{\nu}$ is as required.

Suppose that γ is such that $\nu_\gamma < \alpha$. We need to show that $\nu_\gamma = \sup(D_\gamma \cap \alpha)$. If $\nu_\gamma = \sup(D_\gamma)$ then this is immediate. Otherwise there is some $\vec{\mu} \prec \vec{\nu}$ with $\mu_\gamma > \nu_\gamma$. Since $\gamma > 0$, the definition of the $\prec$-order implies that $D_\gamma \cap \nu_0 + 1 = C_{\mu_\gamma, \gamma \downarrow \mu_\gamma} \cap \nu_0 + 1 \subseteq \nu_\gamma + 1$. Since $\nu_0 > \alpha$ it follows that $\nu_\gamma = \sup(D_\gamma \cap \alpha)$.

To see that the sets D_γ are closed, note that if $\nu \in D_\gamma$ then $D_\gamma \cap \nu$ is a finite union of sets from $\vec{C}$ and hence is closed. Thus the only way that D_γ could fail to be closed would be if $\sup(D_\gamma) < \lambda$ and $\sup(D_\gamma) \notin D_\gamma$. However if $\vec{\nu}$ is as specified by the first paragraph then $\sup(D_\gamma) = \nu_\gamma \in D_\gamma$ for all such γ. $\square$

Proof of lemma 1.24. This lemma follows from Proposition 1.25 and Lemma 1.28. $\square$

Definition 1.29. If $\vec{C}$ is a layered tree sequence and $\alpha < \operatorname{len}(\vec{C})$ then we write $\vec{C}{\restriction}\alpha = \{\, C_{\nu,\beta} : \nu < \alpha \text{ and } \beta < \nu^+ \,\}$; and we write $\vec{C}{\downharpoonleft}\alpha$ for the sequence with domain equal to $\left\{\, (\nu, \beta) \in \operatorname{domain} \vec{C} : \nu \geq \alpha \,\right\}$ which is defined by

$$(\vec{C}{\downharpoonleft}\alpha)_{\nu,\beta} = \begin{cases} C_{\nu,\beta} \setminus \alpha \cup \{\max(C_{\nu,\beta} \cap \alpha)\} & \text{if } C_{\nu,\beta} \cap \alpha \neq \varnothing \\ C_{\nu,\beta} & \text{otherwise.} \end{cases}$$

Notice that if $C_{\nu,\beta} \cap \alpha \neq \varnothing$ then $\left|(\vec{C}{\downharpoonleft}\alpha)_{\nu,\beta} \cap \alpha\right| = 1$.

Proposition 1.30. *If $\vec{C}$ is a layered tree sequence then so are $\vec{C}{\restriction}\alpha + 1$ and $\vec{C}{\downharpoonleft}\alpha + 1$.*

Proof. The proposition is clear for $\vec{C}{\restriction}\alpha + 1$, and we will prove it for $\vec{C}{\downharpoonleft}\alpha + 1$. Let λ be any member of domain$(\vec{C}{\downharpoonleft}\alpha + 1)$, assume as an induction hypothesis that $(\vec{C}{\downharpoonleft}\alpha + 1){\restriction}\lambda$ is a layered tree sequence, and let F_λ be the filter in $P^{\vec{C}}_\lambda$

such that $\vec{C}_\lambda = \operatorname{ext}_\lambda^{\vec{C}\restriction\lambda}(F_\lambda)$. Let $\vec{\nu} \in F_\lambda$ be given by lemma 1.28 so that $\nu_\gamma = \sup(C_{\lambda,\gamma} \cap \alpha+1)$ whenever $\nu_\gamma \leq \alpha$, and set $F'_\lambda = \{\, \vec{\mu} \in F_\lambda : \vec{\mu} \preceq \vec{\nu}\,\} \cup \left\{\, \vec{\mu} \in P_\lambda^{\vec{C}\downharpoonleft\alpha} : \vec{\nu} \preceq \vec{\mu}\,\right\}$. Then F'_λ is a filter in $P_\lambda^{\vec{C}\downharpoonleft\alpha+1}$ satisfying the criteria of Definition 1.22, and $(\vec{C}\downharpoonleft\alpha+1)_\lambda = \operatorname{ext}_\lambda^{\vec{C}\downharpoonleft\alpha+1}(F'_\lambda)$. □

Definition 1.31. If $\vec{C}'$ and $\vec{C}''$ are layered tree sequences such that $\vec{C}'$ has length $\alpha+1$ and $\vec{C}''$ is on the interval (α, λ), then the *join of the trees $\vec{C}''$ and $\vec{C}'$* is the layered tree sequence $\vec{C}$, of length λ such that

1. $\vec{C}\restriction\alpha+1 = \vec{C}'$.
2. $\vec{C}\downharpoonleft\alpha+1 = \vec{C}''$.
3. If $\nu \in (\alpha, \lambda)$ and $\vec{C}''_\nu$ is smoothly joined to $\vec{C}''$ at $\vec{\nu}$, then $\vec{C}_\nu$ is joined to $\vec{C}$ at $\vec{\nu}$.

Maybe a better way of stating this: $\vec{C}\restriction\alpha+1 = \vec{C}'$, and if $\nu > \alpha$ then $\vec{C}_\nu = \operatorname{ext}_\nu^{\vec{C}}(F_\nu)$ where F_ν is the filter in $P_\nu^{\vec{C}}$ generated by the filter F''_ν in $P_\nu^{\vec{C}''}$ such that $\vec{C}''_\nu = \operatorname{ext}_\nu^{\vec{C}''}(F''_\nu)$.

Proposition 1.32. *If $\vec{C}'$ and $\vec{C}''$ are as in definition 1.31 then the join $\vec{C}$ of $\vec{C}'$ and $\vec{C}''$ is well defined and is a layered tree sequence.*

In particular, if $\vec{C}$ is a layered tree sequence with $\operatorname{len}(\vec{C}) > \alpha$ *then $\vec{C}$ is the join of $\vec{C}\restriction\alpha+1$ and $\vec{C}\downharpoonleft\alpha+1$.*

The alternative boxed definition seems to make this straightforward.

Proof. For the first paragraph, notice that the join in clause 3 is smooth: if $\vec{C}''_\lambda$ is smoothly joined to $\vec{C}''$ at $\vec{\nu}$, then $\vec{C}_\lambda$ is also smoothly joined to $\vec{C}$ at $\vec{\nu}$. □

As with the tree sequences in [7], this construction will be used to prove that the forcing $\mathcal{R}_\lambda$ used to construct a layered tree sequence $\vec{C}$ of length λ can be factored as $\mathcal{R}_{\alpha+1} \times \mathcal{R}_{\alpha+1,\lambda}$, where the layered tree sequences $\vec{C}\restriction\alpha+1$ and $\vec{C}\downharpoonleft\alpha+1$ are generic for $\mathcal{R}_{\alpha+1}$ and $\mathcal{R}_{\alpha+1,\lambda}$ respectively.

2 The forcing $\mathcal{R}_\lambda$

We will now begin the definition of the forcing which will be used to add a layered tree sequence $\vec{C}\restriction\kappa$ below κ. This forcing is based on that of [7], and in this section we recall from [7] the basic definitions for the style of iterated forcing used there along with some basic lemmas related to iterations of limit length. Proofs for the results stated in this section may be found in [7].

7/24/07 *** With recent changes this is not so heavily reliant on [7].

The iterated forcing $(\mathcal{R}_\lambda, Q_\lambda : \lambda < \kappa)$ is slightly different from the usual backward Easton forcing. It is a Gitik iteration of Prikry type forcings $(Q_\lambda, \leq$

, $\leq^*$). Being a Prikry type forcing means that $\leq^* \subset \leq$ and that $\leq^*$ is $<\lambda$-closed and has the *Prikry property*: for each sentence σ and condition $q \in Q_\lambda$ there is $q' \leq^* q$ such that q' decides σ. Being an Gitik iteration means that the iterated forcing $\mathcal{R}_\lambda$ has finite support for $\leq$-extensions but Easton support for $\leq^*$-extensions and for additions to the domain of a condition $p \in \mathcal{R}_\lambda$.

We will give precise definition of the iteration $\mathcal{R}_\lambda$ in this section, and of Q_λ is the next section. This section also describes some additional properties of the forcings Q_λ which will transfer to the iterated forcing $\mathcal{R}_\lambda$ and are used in the inductions. One of these properties is, at least in a sense, only notational, but is important to the discussion: As in any iterated forcing, the forcing notion $(Q_\lambda, \leq, \leq^*)$ is a member of the generic extension $V^{\mathcal{R}_\lambda}$ of the ground modle V; however in our forcing both the set Q_λ of conditions and the ordering $\leq^*$ will be members of the ground model V. Only the forcing order $\leq$ will depend on the generic set H_λ.

Definition 2.1. Given the forcing notions Q_α for $\alpha < \kappa$, the Gitik iteration $(\mathcal{R}_\lambda, Q_\lambda : \lambda < \kappa)$ is defined by recursion on λ:

1. A condition of $\mathcal{R}_\lambda$ is a function p such that domain(p) is a Easton support set of Mahlo cardinals less than λ, and $p_\nu \in Q_\nu$ for all $\nu \in \operatorname{domain} p$.
2. If $p', p \in \mathcal{R}_\lambda$ then $p' \leq^* p$ if and only if (a) $\operatorname{domain} p' \supseteq \operatorname{domain} p$ and (b) $p'_\nu \leq^* p_\nu$ for all $\nu \in \operatorname{domain} p$.
3. If $p', p \in \mathcal{R}_\lambda$ then $p' \leq p$ if and only if (a) $\operatorname{domain} p' \supseteq \operatorname{domain} p$, (b) $p' \restriction \nu \Vdash_{\mathcal{R}_\nu} p'_\nu \leq p_\nu$ for all $\nu \in \operatorname{domain} p$, and (c) $\{\, \nu \in \operatorname{domain} p : p'_\nu \not\leq^* p_\nu \,\}$ is finite.

The following definition will be used in section 3.2 as an recursion hypothesis for the definition of Q_λ: in defining Q_λ we will assume that $Q_{\lambda'}$ is suitable for all $\lambda' < \lambda$. Two later definitions, 2.10 and 2.12, at the end of this section will also be used as part of the recursion hypothesis.

Definition 2.2. We will say that a forcing notion $Q_\alpha = (Q_\alpha, \leq^*, \leq)$ as above is *suitable* if it satisfies the following five conditions:

1. $|Q_\alpha| \leq \alpha^+$, and Q_α is trivial unless α is an Mahlo cardinal.
2. The partial order $(Q_\alpha, \leq^*)$ is α-closed in V.
3. Forcing with $(Q_\alpha, \leq)$ over $V^{\mathcal{R}_\alpha}$ preserves α^+.
4. Q_α has the Prikry property: for each formula σ of the forcing language of $\mathcal{R}_{\alpha+1}$ and each $q \in q_\alpha$ there is $q' \leq^* q$ such that $\Vdash_{\mathcal{R}_\alpha} q' \parallel_{\dot{Q}_\alpha} \sigma$.
5. For all $\alpha' < \alpha$ there is a $\leq^*$-dense subset Q^* of Q_α such that if $q \in Q^*$ then for any $p \in \mathcal{R}_\alpha$ and $q' \in Q_\alpha$ such that $p \Vdash_{\mathcal{R}_\alpha} q' \leq q$ we have $p \restriction (\alpha', \alpha) \Vdash_{\mathcal{R}_\alpha} q' \leq q$.

In the remainder of this section we assume that Q_α is suitable for all $\alpha < \lambda$.

Proposition 2.3. *For any $\alpha+1<\lambda$ there is a $\le^*$-dense subset $\mathcal{R}_{\alpha+1,\lambda}$ of $\{\, p\in\mathcal{R}_\lambda : \operatorname{domain}(p)\cap\alpha+1=\varnothing \,\}$ such that the restriction of $\le^{\mathcal{R}_\lambda}$ to $\mathcal{R}_{\alpha+1,\lambda}$ is a member of V. Hence $\mathcal{R}_\lambda\equiv\mathcal{R}_{\alpha+1}\times\mathcal{R}_{\alpha+1,\lambda}$.* $\square$

If $p\in\mathcal{R}_\lambda$ then we write $p\lfloor\lambda'+1$ for $p\restriction(\lambda',\lambda)$. The condition $p\lfloor\lambda'+1$ is not necessarily in $\mathcal{R}_{\lambda'+1,\lambda}$, but by proposition 2.3 there is always a direct extension of $p\lfloor\lambda'+1$ which is in $\mathcal{R}_{\lambda'+1,\lambda}$.

If $H_\lambda\subset\mathcal{R}_\lambda$ is generic then we will write $H_{\lambda'+1}$ for the generic set $H_\lambda\cap\mathcal{R}_{\lambda'+1}$ and $H_{\lambda'+1,\lambda}$ for the generic set $H_\lambda\cap\mathcal{R}_{\lambda'+1,\lambda}$.

Proposition 2.4. *The partial order $(\mathcal{R}_{\mu+1,\lambda},\le^*)$ is $|\mathcal{R}_{\mu+1}|^+$-closed for all $\mu<\lambda$.* $\square$

Corollary 2.5. *Suppose that $\eta<\lambda$, $p\in\mathcal{R}_{\eta+1,\lambda}$ and A is an open subset of $\mathcal{R}_\lambda$. Then there is a condition $p'\le^* p$ in $\mathcal{R}_{\eta+1,\lambda}$ such that for every condition $r\in\mathcal{R}_{\eta+1}$, either $r^\frown p'\in A$ or else there is no $p''\le^* p'$ in $\mathcal{R}_{\eta+1,\lambda}$ such that $r^\frown p''\in A$.* $\square$

The next definition is standard.

Definition 2.6. A forcing $(P,\le)$ is *δ-presaturated* if for every collection $\mathcal{A}$ of antichains with $|\mathcal{A}|<\delta$ there is a $\le$-dense set of conditions p such that for each $A\in\mathcal{A}$ the set of conditions $q\in A$ which compatible with p has size less than δ.

Being δ-presaturated is essentially a local δ-chain condition. It is equivalent to the statement that any set $B\in V^P$ of size less than δ there is a set $B'\supset B$ in V such that $|B'|<\delta$.

Proposition 2.7. *For any limit cardinal λ, the forcing $\mathcal{R}_\lambda$ is λ^+-presaturated. If λ is regular then $\mathcal{R}_\lambda$ is λ-presaturated, and if λ is Mahlo then $\mathcal{R}_\lambda$ satisfies the λ-chain condition.*

Proof. If λ is singular then let $\delta=\operatorname{cf}(\lambda)<\lambda$, let $\langle\,\lambda_\iota:\iota<\delta\,\rangle$ be a continuous, increasing sequence of cardinals, cofinal in λ, with $\lambda_0>\delta$, and let $\mathcal{A}=\{\,A_\iota:\iota<\lambda\,\}$ be a set of maximal antichains. A Condition p as required by Definition 2.6 is found by as the limit of a $\le^*$-descending sequence $\langle\,p_\iota:\iota<\delta\,\rangle$ of conditions in $\mathcal{R}_{\delta+1,\lambda}$, where p_ι is defined using lemma 2.4 so that for any condition $p'\in\mathcal{R}_{\lambda_\iota}$ and any $\nu<\lambda_\iota$, either $p'^\frown(p_\iota\lfloor\lambda_\iota)\le r$ for some $r\in A_\nu$ or else there is no $\bar p\le^*(p_\iota\lfloor\lambda_\iota)$ and $r\in A_\nu$ such that $p'^\frown\bar p\le r$.

Since $(\mathcal{R}_{\delta+1,\lambda},\le^*)$ is δ^+-closed, the limit $p=\bigwedge_{\iota<\delta}p_\iota$ is defined. If $p'\le p$ and $p'\le p''\in A_\nu$ for some $\nu<\lambda$, then there is some $\iota<\delta$ such that $p''\lfloor\lambda_\iota+1\le^* p\lfloor\lambda_\iota+1$. By increasing ι if necessary we can assume that $\iota>\nu$, and it follows that $(p'\restriction\lambda_\iota+1)^\frown(p'\lfloor\lambda_\iota+1)\le p''$. Since $|\mathcal{R}_{\lambda_\iota+1}|<\lambda$, it follows that there are at most λ conditions $p'\in A_\nu$ which are compatible with p.

A similar proof, using a recursion of length ω, shows that $\mathcal{R}_\lambda$ is λ-presaturated if λ is regular. In this case the sequence $\langle\,\lambda_i:i\in n\,\rangle$ of cardinals is defined recursively along with the conditions p_i by setting $\lambda_{i+1}=\sup(\operatorname{domain}p_i)$.

If λ is Mahlo, then a standard proof shows that $\mathcal{R}_\lambda$ has the λ-chain condition. $\square$

Corollary 2.8. *The forcing $\mathcal{R}_\lambda$ preserves all cardinals, and preserves the cofinality of all cardinals δ except possibly those at which Q_δ is nontrivial.* □

Lemma 2.9. *The ordering $\mathcal{R}_\lambda$ has the Prikry property for all ordinals λ. In addition, $\mathcal{R}_{\eta+1,\lambda}$ has the Prikry property for all $\eta+1<\lambda$.* □

Definitions 2.10 and 2.11, along with the associated lemmas, are essentially taken from [7], but the statements are slightly different because we are using layered tree sequences rather than the simple tree sequences used in [7].

Is this what I really want here? I'll wait til I get to the verification that this holds to see if its ok.

Definition 2.10. If M is a model of set theory, $\vec{C}=\langle\,\vec{C}_\nu:\nu<\lambda\,\rangle$ is a layered tree sequence in M of height λ, and Q is a forcing notion in M then we say that Q *extends the tree* $\vec{C}$ if there is a function $f\colon Q\to P^{\vec{C}}_\lambda$ such that whenever $G\subseteq Q$ is generic the set $f``G$ generates a filter in $P^{\vec{C}}_\lambda$ such that (i) $\vec{C}^\frown\langle\lambda,\operatorname{ext}^{\vec{C}}_\lambda(f``G)\rangle$ is a layered tree sequence of height $\lambda+1$ and (ii) the sequence $\operatorname{ext}^{\vec{C}}_\lambda(f``G)$ determines G.

Definition 2.11. If $H_\lambda\subseteq\mathcal{R}_\lambda$ is generic then then we define $\vec{C}(H_\lambda)=\vec{C}\restriction\lambda$, assuming as an induction hypothesis that Q_δ is suitable and extends the tree $\vec{C}\restriction\delta=\vec{C}(H_\delta)$ for each $\delta<\lambda$.

The definition is by recursion on δ. Suppose that $\vec{C}\restriction\delta=\vec{C}(H_\lambda)$ has been defined and that Q_δ is suitable and extends $\vec{C}\restriction\delta$, witnessed by the function f_δ. Then $\vec{C}_\delta=\operatorname{ext}^{\vec{C}\restriction\delta}_\delta(f_\delta``G_\delta)$ where G_δ is the $V[H_\delta]$-generic subset of Q_δ such that $H_{\delta+1}=H_\delta*G_\delta$.

This assumption that Q_δ extends $\vec{C}\restriction\delta$ is the first of the two induction hypotheses which we assume in addition to suitability. The second is the following technical property, which is used in the inductive proof that the forcing notions Q_λ satisfy definition 2.10:

Definition 2.12. We say that $(Q,\le,\le^*)$ is *laudable* if for all $q_0,q_1\in Q$ such that $q_0\Vdash q_1\in\dot{G}$, where $\dot{G}$ is a name for a generic subset of Q, either there is a common $\le^*$-extension q' of q_0 and q_1, or else there is no $q'\le^* q_1$ such that $q'\Vdash q_0\in\dot{G}$.

Proposition 2.13. *If λ is any ordinal and $\mathcal{R}_\alpha\Vdash Q_\alpha$ is laudable for each $\alpha<\lambda$ then $\mathcal{R}_\lambda$ is laudable.*

Proof. Suppose that $p_0,p_1\in\mathcal{R}_\lambda$ and $p_0\Vdash p_1\in\dot{H}$. If p_0 and p_1 have no common $\le^*$-extension then there must be $\alpha\in\operatorname{domain}(p_0)\cap\operatorname{domain}(p_1)$ such that $p_{0,\alpha}$ and $p_{1,\alpha}$ have no common $\le^*$-extension in Q_α. Now suppose that there is $p\le^* p_1$ such that $p\Vdash p_0\in\dot{H}_\lambda$. Then $p_\alpha\le^* p_{1,\alpha}$ and $p\restriction\alpha\Vdash_{\mathcal{R}_\alpha}p_\alpha\Vdash_{Q_\alpha}p_{0,\alpha}\in\dot{G}_\alpha$, contradicting the assumption that Q_α is laudable. □

In the next three sections we define the forcing Q_λ to be used at λ, assuming as an recursion hypothesis that Q_ν is suitable and laudable and extends the tree

$\vec{C}(H_\nu)$ for each $\nu < \lambda$. First we conclude this section by recalling from [7] some notation which is frequently useful in dealing the the forcing $\mathcal{R}_\lambda$:

I may have to move this earlier to use it in the proof of proposition 2.7.

Notation 2.14. If p_1 and p_2 are conditions in $\mathcal{R}_\lambda$ with $\alpha_1 < \alpha_2$ for all $\alpha_1 \in \operatorname{domain}(p_1)$ and $\alpha_2 \in \operatorname{domain}(p_2)$, then we write $p = p_1{}^\frown p_2$ for the condition $p = p_1 \cup p_2$, This notation extends naturally to concatenations of more than two conditions. If $q \in Q_\alpha$ then we will abuse notation by regarding $\langle \alpha, q\rangle$ as a member of $\mathcal{R}_{\alpha+1}$ with domain $\{\alpha\}$, and use this notation in concatenations such as $p_1{}^\frown\langle\alpha, q\rangle{}^\frown p_2$.

3 The definition of $\mathcal{R}_{\lambda+1}$

This concludes the limit case of $\mathcal{R}_\lambda$, and we now turn to the successor case $\mathcal{R}_{\lambda+1}$ for $\lambda < \kappa$. If λ is not Mahlo, then Q_λ is trivial and $\mathcal{R}_{\lambda+1} = \mathcal{R}_\lambda$, so we can assume that λ is Mahlo. We assume as a recursion hypothesis that Q_α has been defined and is suitable and laudable for all $\alpha < \lambda$, and that the layered tree $\vec{C}\restriction\lambda = \vec{C}(H_\lambda)$ is defined for any generic $H_\lambda \subset \mathcal{R}_\lambda$. We will define Q_λ, and show that Q_λ is suitable and laudable and extends $\vec{C}(H_\lambda)$.

3.1 Preliminaries

The conditions of the forcing Q_λ at λ will be pairs $(\mathcal{E}, q)$. The first component $\mathcal{E}$ will be a sequence $\mathcal{E}$ of length $\operatorname{len}(\mathcal{E}) < \lambda^+$ whose members will determine a sequence of sets $\operatorname{val}^H(\mathcal{E}_\gamma) \subseteq \lambda$ in the generic extension $V[H]$. The second component will be a condition q in a Prikry type forcing notion $(Q^{\mathcal{E}}_\lambda, \leq, \leq^*)$ which will add closed unbounded sets $C_{\lambda,\gamma} \subset \operatorname{val}^H(\mathcal{E}_\gamma)$. The forcing Q_λ is essentially a two step iteration in which the first step consists of simply choosing the sequence $\mathcal{E}$ and the second step consists of forcing with $Q^{\mathcal{E}}_\lambda$. This description will be slightly modified, however, by making two conditions $(\mathcal{E}, q)$ and $(\mathcal{E}', q)$ equivalent if $Q^{\mathcal{E}}_\lambda$ and $Q^{\mathcal{E}'}_\lambda$ are identical below q. The following definition makes this precise.

Definition 3.1.
1. The conditions in Q_λ are pairs $(\mathcal{E}, q)$ such that $\mathcal{E}$ is a suitable sequence on λ and $q \in Q^{\mathcal{E}}_\lambda$.
2. If $(\mathcal{E}, q)$ and $(\mathcal{E}', q)$ are in $Q^{\mathcal{E}}_\lambda$ then we say that $(\mathcal{E}, q) \equiv (\mathcal{E}', q)$ if $\operatorname{len}(\mathcal{E}) = \operatorname{len}(\mathcal{E}')$, $q \in Q^{\mathcal{E}}_\lambda \cap Q^{\mathcal{E}'}_\lambda$, and $(Q^{\mathcal{E}}_\lambda/q, \leq, \leq^*) = (Q^{\mathcal{E}'}_\lambda/q, \leq, \leq^*)$, where we write Q/q for the restriction of Q to $\{\, q' \in Q : q' \leq q \,\}$.
3. We say that $(\mathcal{E}', q') \leq (\mathcal{E}, q)$ whenever $q' \leq q$ in $Q^{\mathcal{E}}$ and $(\mathcal{E}', q') \equiv (\mathcal{E}, q')$.
4. We say that $(\mathcal{E}', q') \leq^* (\mathcal{E}, q)$ whenever $q' \leq^* q$ in $Q^{\mathcal{E}}$ and $(\mathcal{E}', q') \equiv (\mathcal{E}, q')$.

Note that clauses 2 and 4, which involve the order $\leq$, are statements in $V^{\mathcal{R}_\lambda}$. The other clauses are statements in V.

The remainder of this subsection we will be dedicated to specifying which sequences $\mathcal{E}$ can appear as the first element of a condition $(\mathcal{E}, q)$ in Q_λ, although doing so will require explaining some aspects of the forcing $Q^{\mathcal{E}}_\lambda$. The next subsection will define the forcings $Q^{\mathcal{E}}_\lambda$, and the proof that these satisfy the Prikry property will take up most of the rest of the section.

To facilitate the discussion of $Q^{\mathcal{E}}_\lambda$ we introduce the following notation: We will write $\mathcal{R}^{\mathcal{E}}_{\lambda+1}$ for $\mathcal{R}_\lambda * Q^{\mathcal{E}}_\lambda$ and $\mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda}$ for $\mathcal{R}_{\nu+1,\lambda} * Q^{\mathcal{E}}_\lambda$

The "suitable sequences" for which $Q^{\mathcal{E}}_\lambda$ is defined will be sequences of the form $\mathcal{E} = \langle (E_\gamma, h_\gamma) : \gamma < \lambda^+ \rangle$. The members $\mathcal{E}_\gamma = (E_\gamma, h_\gamma)$ of this sequence will consist of a set $E_\gamma \subseteq \lambda$ and a function $h_\gamma \colon E_\gamma \to \mathcal{R}^{\mathcal{E}\restriction\gamma}_{\nu+1,\lambda+1}$. The pairs (E_γ, h_γ) will determine sets $\operatorname{val}^H(E_\gamma, h_\gamma) \subset \lambda$ in $V^{\mathcal{R}^{\mathcal{E}\restriction\gamma}_{\nu+1,\lambda+1}}$ consisting roughly of those ordinals $\nu \in E_\gamma$ such that $h_\gamma(\nu)$ is a member of the generic set.

As a special case, the sequence $\mathcal{E}_\kappa$ used at κ will have length κ^+. The pairs $\mathcal{E}_{\kappa,\gamma} = (E_{\kappa,\gamma}, h_{\kappa,\gamma})$ will be choosen so that the collection of sets $\{ E_{\kappa,\gamma} : \gamma < \kappa^+ \}$ generates the repeat point W, and the collection of sets $\{ \operatorname{val}(\mathcal{E}_{\kappa,\gamma}) : \gamma < \kappa^+ \}$ will generate the ultrafilter in the final model. The sequences $\mathcal{E}_\lambda$ used at a cardinals $\lambda < \kappa$, with which we are currently concerned, will have length less than λ^+ but otherwise mirror the behavior of $\mathcal{E}_\kappa$.

First we consider the sets E_γ from the sequence $\mathcal{E}$. Recall that $\mathcal{W}$ is a given coherent sequence of measures in the ground model. We define a subsequence $\mathcal{U}$ of $\mathcal{W}$ which is maximal among those subsequences which satisfy $E_\nu \in \mathcal{U}(\lambda, \beta)$ whenever $\beta \geq \lambda^+ \cdot \nu$.

Definition 3.2.

1. Suppose that $\mathcal{E} = \langle (E_\gamma, h_\gamma) : \gamma < \operatorname{len}(\mathcal{E}) \rangle$ where the sets E_γ form a continuously decreasing sequence of subsets of λ. We define ordinals $\beta^{\mathcal{E}}_{\lambda,\xi}$ by recursion on ξ as follows: suppose that $\beta^{\mathcal{E}}_{\lambda,\xi'}$ has been defined for all $\xi' < \xi$ and that $\lambda^+ \cdot \gamma \leq \xi < \lambda^+ \cdot (\gamma + 1)$, where $\gamma < \operatorname{len}(\mathcal{E})$. Then $\beta^{\mathcal{E}}_{\lambda,\xi}$ is the least ordinal $\beta < o^{\mathcal{W}}(\lambda)$, if there is one, such that $\beta \geq \sup \left\{ \beta^{\mathcal{E}}_{\lambda,\xi'} + 1 : \xi' < \xi \right\}$ and $E_\gamma \in \mathcal{W}(\lambda, \beta)$.
2. We write $o^{\mathcal{E}}(\lambda)$ for the least ordinal ξ such that $\beta^{\mathcal{E}}_{\lambda,\xi}$ is not defined, and we write $\mathcal{U}^{\mathcal{E}}(\lambda, \xi) = \mathcal{W}(\lambda, \beta^{\mathcal{E}}_{\lambda,\xi})$ for each $\xi < o^{\mathcal{E}}(\lambda)$.
3. We write $o^*(\mathcal{E})$ for the least ordinal η such that $o^{\mathcal{E}}(\lambda) \leq \lambda^+ \cdot \eta$.
4. We say that $\mathcal{E}$ is *semi-complete* if $o^*(\mathcal{E}) = \operatorname{len}(\mathcal{E})$, and $\mathcal{E}$ is *complete* if $o^{\mathcal{E}}(\lambda) = \lambda^+ \cdot \operatorname{len}(\mathcal{E})$.

8/3/07 *** Somewhere around here note that I use G_λ both for a generic subset of Q_λ and for the generic subset of $Q^{\mathcal{E}}_\lambda$.

Note that any proper initial segment of a semi-complete sequence $\mathcal{E}$ is complete. The forcing $Q^{\mathcal{E}}_\lambda$ to be defined will have the property that for any sequence $\vec{C}_\lambda$ obtained from a generic subset G of $Q^{\mathcal{E}}_\lambda$, the ordinal $o^*(\mathcal{E})$ associated with $\mathcal{E}$ by clause 3 will be the same as the ordinal $o^*(\vec{C}_\lambda)$ associated with $\vec{C}_\lambda$ by proposition 1.26.

If $\mathcal{E}$ is not semi-complete then $Q^{\mathcal{E}}_\lambda$ will be equal to $Q^{\mathcal{E}\restriction o^*(\mathcal{E})}$, so we usually work only with semi-complete sequences. If a sequence $\mathcal{E}$ has limit length then $\mathcal{E}$ is complete if and only if it is semi-complete. For a sequence $\mathcal{E}$ of successor length the difference between a complete sequence and one which is semi-complete but not complete is the same as the difference in [7] between $o(\lambda) = \lambda^+$ and $o(\lambda) < \lambda^+$: Suppose that $\mathcal{E}$ is a semi-complete sequence of successor length. If $\mathcal{E}$ is not complete then forcing with $Q^{\mathcal{E}}_\lambda$ adds a Prikry-Magidor subset of λ and hence makes λ singular; while if $\mathcal{E}$ is complete then forcing with $Q^{\mathcal{E}}$ not only preserves the regularity of λ, but also preserves any measures $\mathcal{W}(\lambda,\beta)$ such that $\beta > \beta^{\mathcal{E}}_{\lambda,\xi}$ and $E_{\lambda,\xi} \in \mathcal{W}(\lambda,\beta)$ for all $\xi < o^{\mathcal{E}}(\lambda)$.

Before turning to the second component h_γ of the members (E_γ, h_γ) of $\mathcal{E}$, we need to mention some aspects of the definition of the forcing notions $Q^{\mathcal{E}}_\lambda$. Formal statements of these assertions will be given later. First, the partial orders $Q^{\mathcal{E}}_\lambda$ are defined, simultaneously with the concepts in this subsection, by recursion on $\operatorname{len}(\mathcal{E})$; thus we assume here that $Q^{\mathcal{E}\restriction\gamma}_\lambda$ has already been defined for all ordinals $\gamma < \operatorname{len}(\mathcal{E})$.

I think that I should just change the original definition of P_λ to have only Mahlo cardinals. That would be better than changing it here.

Second, a condition $q \in Q^{\mathcal{E}}_\lambda$ is a sequence $q = \langle\, q_\gamma : \gamma < \lambda^+ \,\rangle$, and each member q_γ of the sequence q is a 5-tuple whose first member is a cardinal $\nu_\gamma < \lambda$. The sequence $\vec{\nu}^q = \langle\, \nu_\gamma : \gamma < \lambda^+ \,\rangle$ of these ordinals will be a member of the set P_λ from definition 1.10. If $H_\lambda \subseteq \mathcal{R}_\lambda$ is generic and $q' \leq q$ in the order of $Q^{\mathcal{E}}_\lambda$ in the model $V[H_\lambda]$ then we will have $\vec{\nu}^{q'} \leq \vec{\nu}^q$ in $P^{\vec{C}(H_\lambda)}_\lambda$.

We can now be more explicit about the induction hypothesis stating that $Q_{\lambda'}$ extends the layered tree sequence $\vec{C}\restriction\lambda'$:

Inducton Hypothesis 3.3. *Assume that H_λ is a generic subset of $\mathcal{R}_\lambda$, and for each $\lambda' < \lambda$ and $q \in Q_{\lambda'}$, let $F_{\lambda'} = \{\, \vec{\nu}^q : q \in G_{\lambda'} \,\}$. Then the sets $F_{\lambda'}$ are filters which define a layered tree sequence $\vec{C}(H_\lambda)$ by Definition 1.22, and the function $q \mapsto \vec{\nu}^q$ witness that the forcing notion $Q_{\lambda'}$ extends the layered tree sequence $\vec{C}(H'_\lambda) = \vec{C}(H_\lambda)\restriction\lambda'$.*

Each member $\mathcal{E}_\gamma = (E_\gamma, h_\gamma)$ of the sequence $\mathcal{E}$ represents a $\mathcal{R}^{\mathcal{E}\restriction\gamma}_{\lambda+1}$-name for a subset $\operatorname{val}(E_\gamma, h_\gamma)$ of λ, of which the closed unbounded set $C_{\lambda,\gamma}$ will be a subset. This statement needs to be taken with a grain of salt in the case that γ is a limit ordinal, since in that case restricting the generic set $G_\lambda \subseteq Q^{\mathcal{E}}_\lambda$ to $Q^{\mathcal{E}\restriction\gamma}_\lambda$ does not yield a generic subset of $Q^{\mathcal{E}\restriction\gamma}_\lambda$.

We now begin the formal definition, by recursion on the length $\gamma < \lambda^+$ of $\mathcal{E}$, of three notions: (i) $\mathcal{E}$ is a *suitable sequence* of length γ, (ii) (E, h) is a *simple $\mathcal{E}$-term*, and (iii) the forcing $Q^{\mathcal{E}}_\lambda$.

The logical order of the definition is as follows: Assume that the definition of all three notions is known for sequences $\mathcal{E}'$ with $\operatorname{len}(\mathcal{E}') < \gamma = \operatorname{len}(\mathcal{E})$. We can then define the notion of a suitable sequence $\mathcal{E}$ of length γ, the members (E_ξ, h_ξ) of which are simple $\mathcal{E}\restriction\xi$ terms. The definition of $Q^{\mathcal{E}}_\lambda$ follows, and this definition, in turn, is used for the definition of a simple $\mathcal{E}$-term. Such terms will

then appear as the final members of suitable sequences of length $\gamma+1$, allowing the next stage $\gamma+1$ of the recursion.

These definitions will be presented in a slightly different order, which is dictated by the notation involved. The definition of a simple term is given first, and the definition of a suitable sequence (which is a sequence of simple terms) follows immediately after. Finally the definition of the forcing $Q^{\mathcal{E}}_{\lambda}$ is given in the next subsection.

Definition 3.4. Suppose that $\mathcal{E}$ is a suitable sequence of length γ and that $Q^{\mathcal{E}}_{\lambda}$ has been defined.

1. A *simple* $\mathcal{E}$*-term* is a pair (E, h) such that $E \subseteq \lambda$ and h is a function with domain E such that for each $\nu \in E$ we have $h(\nu) = q{\restriction}\gamma$ for some $q \in Q^{\mathcal{E}}_{\lambda}$ with $I^q = \{\gamma\}$ and $\nu^q_\gamma = \nu$.

2. Suppose that $H \subset \mathcal{R}^{\mathcal{E}}_{\lambda+1}$ is generic and (E, h) is a simple $\mathcal{E}{\restriction}\gamma$-term for some $\gamma \leq \operatorname{len}(\mathcal{E})$. If we write $H = H_\lambda * G_\lambda$ then

$$\operatorname{val}^H(E, h) = \{\, \nu \in E : h(\nu){\restriction}\lambda \in H_\lambda \;\;\&\;\; \exists q \in G_\lambda \, h(\nu)_\lambda = q{\restriction}\gamma \,\} .$$

Note that clause 1 asserts that the sequence $\vec{\nu}^{h(\nu)_\lambda}$ is a sequence $\vec{\nu}$ of length γ such that $\nu_\xi = \nu$ if $\xi < \gamma$ and $\xi \propto_\nu \gamma$, and $\nu_\xi = 0$ otherwise. In the case that γ is a successor ordinal it would be equivalent to specify $I^q = \{\gamma - 1\}$.

If $\operatorname{len}(\mathcal{E}) = \gamma$ then clause 2 specifies the obvious $\mathcal{R}^{\mathcal{E}}_{\lambda+1}$-term corresponding to the pair (E, h). If $\gamma < \operatorname{len}(\mathcal{E})$ then the situation is more subtle, since it is not clear that a generic subset of $\mathcal{R}^{\mathcal{E}}_{\lambda+1}$ can be restricted to yield a generic subset H' of $\mathcal{R}^{\mathcal{E}\restriction\gamma}_{\lambda+1}$. In the case that γ is a successor ordinal then we will eventually see that such a restriction H' does exist, and that $\operatorname{val}^H(E, h) = \operatorname{val}^{H'}(E, h)$.

In the case that γ is a limit ordinal and $\gamma < \operatorname{len}(\mathcal{E})$ there is no generic subset of $\mathcal{R}^{\mathcal{E}\restriction\gamma}_{\lambda+1}$ in $V[H]$, where H is a generic subset of $\mathcal{R}^{\mathcal{E}}_{\lambda+1}$. For the reason we only work with simple $\mathcal{E}{\restriction}\gamma$-terms which are diagonal intersections of simple $\mathcal{E}{\restriction}\gamma'$-terms for $\gamma' < \gamma$. The following notation makes this more precise.

Notation 3.5.

1. Suppose that (E, h) and (E', h') are simple $\mathcal{E}{\restriction}\gamma$-terms and $\mathcal{E}{\restriction}\gamma'$-terms, respecitively. Then we say that $(E, h) \subseteq (E', h')$ if $E \subseteq E'$ and $h(\nu){\restriction}\gamma' \leq^* h'(\nu)$ for all $\nu \in E'$.

2. If $\mathcal{E} = \langle\, (E_{\gamma'}, h_{\gamma'}) : \gamma' < \gamma \,\rangle$, where $(E_{\gamma'}, h_{\gamma'})$ is a simple $\mathcal{E}{\restriction}\gamma'$-term, then we write $\Delta(\mathcal{E})$ for the *diagonal intersection* of the sequence $\mathcal{E}$, that is, the simple term (E, h) such that

$$E = \mathop{\Delta}_{\gamma'<\gamma} E_{\gamma'} \text{ and}$$

$$h(\nu){\restriction}\gamma' = \bigwedge \{\, h_{\gamma''}(\nu){\restriction}\gamma' : \gamma' \leq \gamma'' < \gamma \;\;\&\;\; \gamma'' \propto_\nu \gamma \,\}$$

for each $\nu \in E$ and $\gamma' < \gamma$.

Note that $(E,h) \subseteq (E',h')$ implies $\mathrm{val}^H(E,h)$, but the converse implication does not hold since clause 1 requires $h(\nu)\restriction\gamma \leq^* h'(\nu)$, instead of $h(\nu)\restriction\gamma \leq h'(\nu)$. It is worth noting, however, that this comment only applies to $h\restriction\lambda$. The fact that $\vec{\nu}^{h(\nu)_\lambda} = \vec{\nu}^{h'(\nu)_\lambda}$ will ensure that $h(\nu)_\lambda \leq h'(\nu)_\lambda$ if and only if $h(\nu)_\lambda \leq^* h'(\nu)_\lambda$.

The use of $\leq^*$ ensures that the set of simple $\mathcal{E}$-terms is closed under intersections of $\subseteq$-descending sequences $\langle\, (E_\nu, h_\nu) : \nu < \xi \,\rangle$ with $\xi < \min(E_0)$, and that the diagonal intersections called for in the following definition exist.

Definition 3.6. A *suitable sequence* $\mathcal{E}$ *on* λ of length $\gamma < \lambda^+$ is a semi-complete sequence $\langle\, (E_\xi, h_\xi) : \xi < \gamma \,\rangle$ such that for each $\xi < \gamma$,

1. (E_ξ, h_ξ) is a simple $\mathcal{E}\restriction\xi$-term.
2. If ξ is a limit ordinal then $(E_\xi, h_\xi) = \Delta\, \mathcal{E}\restriction\xi$.
3. If ξ is a successor ordinal then $(E_\xi, h_\xi) \subseteq (E_{\xi-1}, h_{\xi-1})$.

3.2 Definition of $Q^{\mathcal{E}}_\lambda$

We are now ready to give the definition of the partial order $Q^{\mathcal{E}}_\lambda$ where $\mathcal{E}$ is a suitable sequence of length γ. As was pointed out earlier, we assume that $Q^{\mathcal{E}'}_\lambda$ and the notion of a simple $\mathcal{E}'$-term have been defined for all sequences $\mathcal{E}'$ of length less than γ, and the notion "$\mathcal{E}'$ is suitable" has been defined for all sequences of length $\mathrm{len}(\mathcal{E}') \leq \gamma$.

The members q of $Q^{\mathcal{E}}_\lambda$ will have the form

$$q = \langle\, q_\iota : \iota < \lambda^+ \,\rangle \text{ where } q_\iota = (\nu_\iota, \beta_\iota, A_\iota, g_\iota, f_\iota).$$

The coordinates q_ι are based on the forcing Q_λ used in [7] to add a single new closed unbounded subset of λ. In the case when $\mathrm{len}(\mathcal{E}) > \iota + 1$ or $\mathrm{len}(\mathcal{E}) = \iota + 1$ and $\mathcal{E}$ is complete, q_ι will be essentially the same as the conditions in the forcing for the case $o(\lambda) = \lambda^+$ of [7], modified by requiring that $g_\iota(\nu)$ be a member of $\mathcal{R}^{\mathcal{E}\restriction\iota}_{\nu+1,\lambda+1}$ instead of $\mathcal{R}_{\nu+1,\lambda}$ and by including specifications which ensure that $C_{\lambda,\iota} \subset \mathrm{val}(E_\iota, h_\iota)$. When $\iota + 1 = \mathrm{len}(\mathcal{E})$ and $\mathcal{E}$ is not complete, say $o^{\mathcal{E}}(\lambda) = \lambda^+ \cdot \iota + \xi$ for some ξ with $0 < \xi < \lambda^+$, then β_ι will take the constant value ξ on a dense subset of $Q^{\mathcal{E}}_\lambda$, and the requirements on the remaining coordinates $(\nu_\iota, A_\iota, g_\iota, f_\iota)$ are essentially the same as those for the case $o(\lambda) = \xi < \lambda^+$ of [7].

The case $\iota \geq \mathrm{len}(\mathcal{E})$ corresponds to the case $o(\lambda) = 0$ of [7]. In this case $(\nu_\iota, \beta_\iota, A_\iota, g_\iota, f_\iota) = (\nu_\iota, 0, \varnothing, \varnothing, \varnothing)$, and the fact that $\vec{\nu} \in P_\lambda$ implies that $\{\, \iota : \nu^q_\iota \neq 0 \,\}$ is bounded in λ^+, so q can be regarded as a member of H_{λ^+}. While ν_ι need not be 0 for all $\iota \geq \mathrm{len}(\mathcal{E})$, the definition of the order $\leq$ on $Q^{\mathcal{E}}_\lambda$ will state that if $q' \leq q$ then $\nu^{q'}_\iota = \nu^q_\iota$ for all $\iota \geq \mathrm{len}(\mathcal{E})$.

We can also now give a formal definition of the projection refered to previously:

Definition 3.7. If $q \in Q^{\mathcal{E}}_\lambda$ and $\gamma < \mathrm{len}(\mathcal{E})$ then we write $q|\gamma$ for the sequence $q' = q\restriction\gamma {}^\frown \langle\, (\nu^q_\iota, 0, \varnothing, \varnothing, \varnothing) : \gamma \leq \iota < \lambda^+ \,\rangle \in Q^{\mathcal{E}\restriction\gamma}_\lambda$.

We are now ready to begin the formal definition of $Q^{\mathcal{E}}_\lambda$. First we define $Q^{\mathcal{E}}_\lambda$ in the easy cases, when $\operatorname{len}(\mathcal{E}) = 0$ or $\operatorname{len}(\mathcal{E})$ is a limit ordinal.

Definition 3.8. A sequence q is in $Q^{\varnothing}_\lambda$ if and only if $q_\iota = (\nu^q_i, 0, \emptyset, \emptyset, \emptyset)$ for all $\iota < \lambda^+$ and $\vec{\nu}^q = \langle\, \nu^q_\iota : \iota < \lambda^+ \,\rangle \in P_\lambda$. The orderings $\leq$ and $\leq^*$ of $Q^{\varnothing}_\lambda$ are trivial: $q' \leq q$ and $q' \leq^* q$ each hold if and only if $q' = q$.

Definition 3.9. If $\operatorname{len}(\mathcal{E})$ is a limit ordinal then $q \in Q^{\mathcal{E}}_\lambda$ if and only if $q = q|\gamma$ and $q|\gamma \in Q^{\mathcal{E}\restriction\gamma}_\lambda$ for all $\gamma < \operatorname{len}(\mathcal{E})$. The direct extension ordering on $Q^{\mathcal{E}}_\lambda$ is defined by $q' \leq^* q$ in $Q^{\mathcal{E}}$ if and only if for all $\gamma < \operatorname{len}(\mathcal{E})$ we have $q'|\gamma \leq^* q|\gamma$ in $Q^{\mathcal{E}\restriction\gamma}_\lambda$.

If q' and q are in $Q^{\mathcal{E}}_\lambda$ then $q' \leq q$ if there is some $\gamma < \operatorname{len}(\mathcal{E})$ such that $\vec{\nu}^{q'}\lfloor\gamma = \vec{\nu}^q\lfloor\gamma$ and $q'|\gamma' \leq q|\gamma'$ in $Q^{\mathcal{E}\restriction\gamma'}_\lambda$ for all $\gamma \leq \gamma' < \operatorname{len}(\mathcal{E})$.

Now we turn to the definition of $Q^{\mathcal{E}}$ when $\mathcal{E}$ is a semi-complete sequence of successor length $\eta + 1$, so that $o^{\mathcal{E}}(\mathcal{E}) = \lambda^+ \cdot \eta + \xi$ for some ordinal ξ with $0 < \xi \leq \lambda^+$. The basic idea is to combine the forcing $Q^{\mathcal{E}\restriction\eta}_\lambda$ to add $C_{\lambda,\iota}$ for $\iota < \eta$ and the forcing of [7] to add $C_{\lambda,\eta}$; however the combination will be equivalent to the suggested two step iteration only in the case that η is a successor ordinal.

It is convenient to use the following notation also in the case when $\xi = 0$, that is, when $\mathcal{E}\restriction\eta$ is a complete sequence of length η but $\mathcal{E}$ is not a semicomplete sequence of length $\eta + 1$. This is potentially ambiguous when η is a successor ordinal, as then $o^{\mathcal{E}}(\lambda)$ could be written either as $\eta \cdot \lambda^+ + 0$ or as $(\eta - 1) \cdot \lambda^+ + \lambda^+$, so that $\xi = o^{\mathcal{E}}(\lambda)$ could be either 0 or λ^+. However the value of η, and hence of $o^{\mathcal{E}}(\lambda)$, will be clear from the context.

Notation 3.10.
1. Suppose that $o^{\mathcal{E}}(\lambda) = \lambda^+ \cdot \eta + \xi$ where $0 \leq \xi \leq \lambda^+$. Then we write $\bar{o}^{\mathcal{E}}(\lambda) = \xi$, and if $\gamma < \xi$ then we write $\overline{U}^{\mathcal{E}}(\lambda, \gamma) = \mathcal{U}^{\mathcal{E}}(\lambda, \lambda^+ \cdot \eta + \gamma)$.
2. If $\nu < \lambda$ then we say that $\bar{o}^{\mathcal{E}}(\nu) = \xi$ if (i) $\xi \propto_\nu \bar{o}^{\mathcal{E}}(\lambda)$, (ii) $\eta{\downarrow}\nu$ is defined, (iii) $\mathcal{E}{\downarrow}\nu$ is semi-complete on ν (or $\xi = 0$ and $\mathcal{E}{\downarrow}\nu$ is complete of length $\eta{\downarrow}\nu$), and (iv) $\xi{\downarrow}\nu = \bar{o}^{\mathcal{E}\downarrow\nu}(\nu)$.
3. If $\beta < \bar{o}^{\mathcal{E}}(\lambda)$ then we will write $\bar{o}^{\mathcal{E},\beta}(\nu) = \xi$ if $\bar{o}^{\mathcal{E}}(\nu) = \xi$ in $\operatorname{ult}(V, \overline{U}^{\mathcal{E}}(\lambda, \beta))$. Note that $\bar{o}^{\mathcal{E},\beta}(\lambda) = \beta$.

If $\mathcal{E}$ is complete, so that $\bar{o}^{\mathcal{E}}(\lambda) = \lambda^+$, then $\bar{o}^{\mathcal{E}}(\nu)$ is normally undefined since $\xi \propto_\nu \lambda^+$ does not hold. However definition still makes sense in the special case $\xi < \lambda$, provided we replace the requirement $\xi \propto_\nu \lambda^+$ with the requirement $\xi < \nu$, and we will write $\bar{o}^{\mathcal{E}}(\nu) = \xi$ in this case.

The function $\bar{o}^{\mathcal{E}}(\nu)$ is not defined for every $\nu < \lambda$, but it is uniquely determined in the cases it is defined. Also, if $\bar{o}^{\mathcal{E}}(\lambda) < \lambda^+$ then $\{\, \nu : \bar{o}^{\mathcal{E}}(\nu) = \xi \,\} \in \overline{U}^{\mathcal{E}}(\lambda, \xi)$ for each $\xi < \bar{o}^{\mathcal{E}}(\lambda)$.

Here is the complete definition of the conditions for successor length. Recall that the set of conditions of $Q^{\mathcal{E}}_\lambda$ and the direct extension order $\leq^*$ on $Q^{\mathcal{E}}_\lambda$ are defined in V, while the forcing order $\leq$ of $Q^{\mathcal{E}}_\lambda$ is defined in $V[H_\lambda]$.

Definition 3.11. If $\operatorname{len}(\mathcal{E}) = \eta+1$ then a sequence q is in $Q^{\mathcal{E}}_{\lambda}$ if $q|\eta \in Q^{\mathcal{E}\restriction\eta}_{\lambda}$, $\vec{\nu}^q = \langle \nu_\iota : \iota < \lambda^+ \rangle \in P_\lambda$, $q_\gamma = (\nu_\gamma, 0, \emptyset, \emptyset, \emptyset)$ for all $\gamma > \eta$, and $q_\eta = (\nu_\eta, \beta_\eta, A_\eta, g_\eta, f_\eta)$ satisfies the following three conditions:

1. $\beta_\eta < \min(\bar{o}^{\mathcal{E}}(\lambda) + 1, \lambda^+)$.
2. (A_η, g_η) is a simple $\mathcal{E}\restriction\eta$-term such that $(A_\eta, g_\eta) \subseteq (E_\eta, h_\eta) = \mathcal{E}_\eta$, $(A_\eta, g_\eta) \subseteq \Delta_{\gamma<\eta}(A_\gamma, g_\gamma)$, and $A_\eta \in \bigcap_{\gamma<\beta_\eta} \bar{U}^{\mathcal{E}}(\lambda, \gamma)$.
3. For all $\nu \in A_\eta$ the following conditions hold:
   (a) $\nu \leq f_\eta(\nu) < \lambda$.
   (b) If we write $q' = g(\nu)_\lambda$ then $\vec{\nu}^{q'} = \operatorname{add}(\vec{\nu}^q, \eta, f_\eta(\nu))$,
   (c) If $f_\eta(\nu) > \nu$ then $g_\eta(\nu)\restriction\lambda$ forces in $\mathcal{R}_{\nu+1,\lambda}$ that there is a condition $(\bar{\mathcal{E}}, \bar{q}) \in \dot{G}_{f_\eta(\nu)}$ with $\bar{\mathcal{E}}\restriction(\eta{\downarrow}f_\eta(\nu)) + 1 = \mathcal{E}{\downarrow}f_\eta(\nu)$ and $\nu^{\bar{q}}_{\eta\downarrow f_\eta(\nu)} = \nu$.
   (d) $g_\eta(\nu) \Vdash_{\mathcal{R}^{\mathcal{E}\restriction\eta}_{\nu+1,\lambda+1}} f_\eta(\nu) \in \operatorname{val}(E_\eta, h_\eta)$.

8/8/07 *** I'm not sure I like 3d, especially in the case η is a limit ordinal. Maybe be explicit: $\nu \in E_\eta$ and $g_\eta(\nu) \leq h_\eta(\nu)$. In any case, this probably calls for a comment.

7/27/07 — Note that there need be no relation between $g^q_\eta(\nu)$ and $q|\lambda$. By Definition 3.17(3a) $g^q_\eta(\nu)$ replaces $q|\lambda$.

It might seem that this should not be true in the Prikry-Magidor case, where q_η is preserved in all extensions. However $q|\eta$ is not preserved, as $\mathcal{E}\restriction\eta$ is complete.

This definition combines the two cases when $\mathcal{E}$ is complete and when $\mathcal{E}$ is not complete. In the case that $\mathcal{E}$ is not complete the set of conditions q with $\beta^q_\eta = \bar{o}^{\mathcal{E}}(\lambda)$ will be open and dense in both of the orders $\leq$ and $\leq^*$. We will later discover that if q is a condition in this set such that f^q_η is the identity, then q will force that $C_{\lambda,\eta} \setminus \nu^q_\eta$ is a Prikry-Magidor-Radin set with $C_{\lambda,\eta} \setminus \nu \subseteq \operatorname{val}^{\mathcal{E}}(A^q_\eta, g^q_\eta)$. This set $C_{\lambda,\eta}$ will change the cofinality of λ to $\operatorname{cf}(\beta)$ if $\operatorname{cf}(\beta) < \lambda$ and to ω otherwise. If q is in this set but f^q_η is not the identity, then q will still force that a new Prikry-Magidor set C is added, but in this case

$$C_{\lambda,\gamma} \setminus \nu^q_\eta = C \cup \bigcup_{\nu\in C} \left(\{f^q_\eta(\nu)\} \cup C_{f^q(\nu),\eta\downarrow f^q(\nu)} \setminus \nu\right) \subset C.$$

Conditions 3(c,d) will ensure that $\nu \in C_{f^q(\nu'),\eta\downarrow f^q(\nu)}$ for each $\nu \in C$, and that $C_{\lambda,\eta} \subset \operatorname{val}(E_\eta, h_\eta)$.

If $\mathcal{E}$ is complete then the set of conditions q such that $\beta^q_\eta = 0$ will be $\leq$-dense. The forcing on this dense subset, which we will call $P^{\mathcal{E}}_{\lambda}$, is similar to the forcing used by Gitik in [1] in the case $o(\lambda) = \lambda$. Gitik defined $\leq^*$ in this forcing to be the same as $\leq$; however our definition of $\leq^*$ will use conditions with $\beta^q_\eta > 0$. As a consequence our order $\leq^*$ will satisfy completeness properties where Gitik's satisfied only distributivity.

We now define the direct extension ordering $\leq^*$.

Definition 3.12. If $\operatorname{len}(\mathcal{E}) = \eta + 1$ and q' and q are in $Q^{\mathcal{E}}_{\lambda}$ then $q' \leq^* q$ if

1. $q'|\eta \leq^* q|\eta$ in $Q^{\mathcal{E}\restriction\eta}_{\lambda}$.
2. $\nu^{q'}_{\eta} = \nu^{q}_{\eta}$.
3. $\beta^{q'}_{\eta} \geq \beta^{q}_{\eta}$.
4. For all $\nu \in A^{q'}_{\eta}$,
   (a) $\beta^{q}_{\eta} \propto_{\nu} \beta^{q'}_{\eta}$.
   (b) If $\nu \in A^{q}_{\eta}$ then $f^{q'}_{\eta}(\nu) = f^{q}_{\eta}(\nu)$ and $g^{q'}_{\eta}(\nu) \leq^* g^{q}_{\eta}(\nu)$.
   (c) If $\nu \notin A^{q}_{\eta}$ then $\bar{o}^{\mathcal{E},\beta^{q'}_{\eta}}(\nu)$ is defined, $\beta^{q'}_{\eta} > \bar{o}^{\mathcal{E},\beta^{q'}_{\eta}}(\nu) \geq \beta^{q}_{\eta}$, and $q{\downarrow}\nu \in Q^{\mathcal{E}\downarrow\nu}_{\nu}$.

Following is some stuff that should go somewhere, and some of which may already be somewhere. This needs to be reorganized.

Definition 3.13. 1. If $q \in Q^{\mathcal{E}}_{\lambda}$ and $\nu < \lambda$ then q/ν is the condition obtained by replacing A^{q}_{γ} with $A^{q}_{\gamma} \setminus \nu$ for all $\gamma < \operatorname{len}(\mathcal{E})$.

2. We say that *the extension $q' \leq^* q$ is above ν* if $A^{q'}_{\gamma} \cap \nu = A^{q}_{\gamma} \cap \nu$, and in addition $g^{q'}_{\gamma}(\nu') \leq^* g^{q}_{\gamma}(\nu')$ is above ν for each $\nu' \in A^{q}_{\gamma} \cap \nu$.

Proposition 3.14. *Suppose that $q_{\nu'} \leq^* q_{\nu}$ for $\nu < \nu' < \xi \leq \lambda$. Then there is q such that $q/\nu \leq^* q_{\nu}$ for all $\nu < \xi$.*

If $q_{\nu'} \leq^* q_{\nu}$ is above ν for $\nu < \nu' < \xi$ then $q \leq^* q_{\nu}$ can be above ν.

In another direction, if $\xi < \lambda$ then (by taking q/ξ for q) we can have $q \leq^* q_{\nu}$ for all $\nu < \xi$.

Proof. The sticky point is Clause 3.12(4a), which requires that $\beta^{q_{\nu}}_{\gamma} \propto_{\nu'} \beta^{q}_{\gamma}$ for all $\gamma < \operatorname{len}(\mathcal{E})$ and $\nu' < \lambda$. This can be arranged by choosing β^{q}_{γ} so that $\langle\, \beta^{q_{\nu}}_{\gamma} : \nu < \xi \,\rangle$ is definable (without parameters) from β^{q}_{γ}. Cf Gitik's arguments in his not-sch stuff. $\square$

The forcing $(Q^{\mathcal{E}}_{\lambda}, \leq^*)$ doesn't have infinums: if $\langle\, q_{\nu} : \nu < \xi \,\rangle$ is a $\leq^*$-decreasing sequence then in general there is no greatest lower bound, even if $\xi < \lambda$. I will be writing $\bigwedge_{\nu<\xi} q_{\nu}$ to designate some $\leq^*$-lower bound, without being specific about which one to use.

Notice that, as pointed out previously, if $\mathcal{E}$ is not complete then there is a $\leq^*$-dense set of conditions q such that β^{q}_{η} has its maximum value $\bar{o}^{\mathcal{E}}(\lambda)$. If q is in this set then $q' \leq^* q$ if and only if $A^{q'} \subset A^{q}$ and clause 4b holds. However it is useful for technical reasons to allow $\beta^{q}_{\eta} < \bar{o}^{\mathcal{E}}(\lambda)$. In clause 4c, for example, the definition of Q_{ν} allows the possibility that $\beta^{q\downarrow\nu}_{\eta\downarrow\nu} < \xi^{\mathcal{E}\downarrow\nu} < \nu^{+}$ and $\bar{o}^{\mathcal{E}\downarrow\nu}(\nu) > \beta^{q\downarrow\nu}_{\eta\downarrow\nu} = \beta^{q}_{\nu}{\downarrow}\nu$.

We now turn to the definition of the forcing order $\leq$ on $Q^{\mathcal{E}}_{\lambda}$, which involves describing extensions $q' < q$ which add new members to the set $C_{\lambda,\eta}$. Recall that $\operatorname{add}(\vec{\nu},\gamma,\nu)$ was defined in definition 1.15.

> Note that for fixed q and η and sufficiently large ν, by proposition 1.12 we have $\operatorname{add}(q,\eta,\nu) = (g^q_\eta)_\lambda\restriction\eta^\frown q\lfloor\eta$ if $\nu \in A^q_\eta$ and $\operatorname{add}(q,\eta,\nu) = (h_\eta)_\lambda\restriction\eta^\frown q\lfloor\eta$ if $\nu \notin A^q_\eta$.
>
> I'm nervous about making this the definition because of the need for homogeneity; however this can be used throughout the proof of the Prikry property.

> This emended version of the definition needs checking. It would be nice if it could be made a bit cleaner, too. This might involve changing the definition of $\|$ or even the h_η in the definition of a simple term.
>
> Clause 1 should probably have $(q'\|\eta)\restriction\eta = h_\eta(\nu)_\lambda\restriction\eta$. Below η, g comes in as $(g^q_\nu\restriction\eta)\downarrow\nu \in G_\nu$.

> 6/27/07 — Actually, I believe the assumption that $\operatorname{add}(\vec{\nu}^q,\eta,\nu) \prec \vec{\nu}^q$ in $P^{\vec{C}}_{\lambda}$ implies (for the case $\nu \in A^q_\eta$) that $\operatorname{add}(\vec{\nu},\eta,\nu) = (g^q_\eta)_\lambda\restriction\eta^\frown q\lfloor\eta$. ***Seems to be correct.

Definition 3.15. Suppose that $\operatorname{len}(\mathcal{E}) = \eta+1$, $\vec{\nu} \in P_\nu$, and $\vec{\nu}\downarrow\nu$ exists. Then we write $t^{\mathcal{E}}_\nu(\vec{\nu})$ for the condition $q \in Q^{\mathcal{E}}_\lambda$ with $\vec{\nu}^q = \operatorname{add}(\vec{\nu},\eta,\nu)$ which is defined by

$$\begin{aligned} q\restriction\eta &= h_\eta(\nu)_\lambda \\ q_\eta &= (\nu, 0, \emptyset, \emptyset, \emptyset) \\ q_\iota &= (\nu_\iota, 0, \varnothing, \varnothing, \varnothing) && \text{for all } \iota \geq \eta. \end{aligned}$$

We will write $t^{\mathcal{E}}_\nu$ for $t^{\mathcal{E}}_\nu(\vec{0})$, and we will write $t^{\mathcal{E}}(q)$ for $t^{\mathcal{E}}(\vec{\nu}^q)$.

Note that $t^{\mathcal{E}}_\nu(\vec{\nu})$ is be the $\leq^*$-weakest condition q such that $\vec{\nu}^q = \operatorname{add}(\vec{\nu},\eta,\nu)$. It will be seen in Definition 3.17 that for two conditions q and q' with $\vec{\nu}^q = \vec{\nu}^{q'}$ we have $q \leq q'$ if and only if $q \leq^* q'$; hence $t^{\mathcal{E}}(\vec{\nu})$ is also the $\leq$-weakest such condition.

If $\mathcal{E}$ is complete then the conditions of form $t^{\mathcal{E}}_\nu(\vec{\nu})$ will be dense in $Q^{\mathcal{E}}_\lambda$. The restriction of $Q^{\mathcal{E}}$ to such conditions is essentially the same as Gitik's forcing from [1] in the case $o(\lambda) = \lambda$ (or, rather, to its adaptation to $o(\lambda) = \lambda^+$).

Definition 3.16. Assume that $\vec{\nu}' = \operatorname{add}(\vec{\nu}^q,\eta,\nu)$ is defined and $\operatorname{add}(\vec{\nu}^q,\eta,\nu) \prec \vec{\nu}^q$ in $P^{\vec{C}}_\lambda$. Then $\operatorname{add}(q,\eta,\nu)$ is the condition $q' \in Q^{\mathcal{E}}_\lambda$ with $q'\lfloor\eta+1 = q\lfloor\eta+1$ such that

1. if $\nu \in A^q_\eta$ then

$$\begin{aligned} q'\restriction\eta &= g^q_\eta(\nu)_\lambda\restriction\eta, \\ q'_\eta &= (f^q_\eta(\nu), A^q_\eta \setminus f^q_\eta(\nu)+1, \beta^q_\eta, f^q_\eta, g^q_\eta), \end{aligned}$$

2. and if $\nu \notin A^q_\eta$ then $q'\restriction\eta+1 = t^{\mathcal{E}}_\nu(\vec{\nu})\restriction\eta+1$.

Note that in the important case that $\operatorname{len}(\mathcal{E}) = \eta + 1$ this means that for $\nu \notin A^q_\eta$ we have $q' = t^{\mathcal{E}}_\nu(\vec{q})$.

Definition 3.17. If $\operatorname{len}(\mathcal{E}) = \eta + 1$ then the forcing ordering $\leq$ of $Q^{\mathcal{E}}_\lambda$ is the least transitive relation satisfying the following five conditions:

1. $q' \leq q$ whenever $q' \leq^* q$.
2. $q' \leq q$ whenever $q'|\eta \leq q|\eta$ in $Q^{\mathcal{E}\restriction\eta}_\lambda$ and $q'\lfloor\eta = q\lfloor\eta$.
3. If $\nu \in A^q_\eta$ then $\operatorname{add}^{\mathcal{E}}(q, \eta, \nu) \leq q$ whenever the following conditions are satisfied:
   (a) $(\mathcal{E}, q'')\downarrow\nu \in G_\nu$, where q'' is the condition obtained from q by replacing β^q_η with $\beta^{q''}_\lambda = \bar{o}^{\mathcal{E}}(\nu)$
   (b) $g^q_\eta(\nu)\restriction\lambda \in H_\lambda$.
4. If $\nu \notin A^q_\eta$ and $q' = \operatorname{add}^{\mathcal{E}}(q, \eta, \nu) = t^{\mathcal{E}}_\nu(\vec{\nu}^q)$ then $q' \leq q$ whenever
   (a) $(\mathcal{E}, q)\downarrow\nu \in G_\nu$.
   (b) $h_\eta(\nu)\restriction\lambda \in H_\lambda$.
   (c) $\nu \in E_\eta$.
   (d) $\bar{o}^{\mathcal{E}\downarrow\nu}(\nu) = \beta^q_\eta\downarrow\nu$.
5. $t^{\mathcal{E}}_{\nu'}(\vec{\nu}) \leq t^{\mathcal{E}}_\nu(\vec{\nu})$ whenever, setting $\bar{\eta} = \eta\downarrow\nu'$,
   (a) There is a sequence $\bar{\mathcal{E}}$ with $\bar{\mathcal{E}}\restriction\bar{\eta}{+}1 = \mathcal{E}\downarrow\nu'$ so that $(\bar{\mathcal{E}}, t^{\mathcal{E}}_\nu(\vec{\nu})\downarrow\nu') \in G_{\nu'}$.
   (b) $h_\eta(\nu')\restriction\lambda \in H_\lambda$.
   (c) $\nu' \in E_\eta$.

Note: In clause 4, it would be incorrect to write "$\bar{o}^{\mathcal{E}}(\nu) = \beta^q_\nu$" because it could be (in fact, would normally be) that $\bar{o}^{\mathcal{E}}(\lambda) = \lambda^+$, in which case $\bar{o}^{\mathcal{E}}(\nu)$ is not defined.

Clause 3 describes the Prikry-Magidor type forcing used for incomplete sequences, which is used as a preparation forcing so that the forcing of clause 5 has the required distributivity properties. If $\mathcal{E}$ is incomplete and q is a member of the $\leq^*$-dense set of conditions with $\beta^q_\eta = \bar{o}^{\mathcal{E}}(\lambda)$ then the ordering of $Q^{\mathcal{E}}$ below q only uses clauses 1–3.

If $\mathcal{E}$ is complete then by clause 4 every condition in $Q^{\mathcal{E}}_\lambda$ can be extended to a condition of the form $t^{\mathcal{E}}_\nu(\vec{\nu})$. Hence the set of conditions q with $\beta^q_\eta = 0$ is dense in $Q^{\mathcal{E}}_\lambda$; however conditions with $\beta^q_\eta > 0$ are needed for the direct extension order $\leq^*$.

As a first step towards proving that the Induction Hypothesis 3.3 holds for Q_λ, we need to verify that $F_\lambda = \{\, \vec{\nu}^q : q \in G_\nu \,\}$. is a filter in $P^{\vec{C}\restriction\lambda}_\lambda$.

Lemma 3.18. *If $q' \le q$ in $Q^{\mathcal{E}}_\lambda$ then $\vec{\nu}^{q'} \preccurlyeq \vec{\nu}^q$ in $P^{\vec{C}}_\lambda$.*

Proof. We prove the lemma by induction on $\operatorname{len}(\mathcal{E})$. If $\operatorname{len}(\mathcal{E}) = 0$ then $q' = q$, so the conclusion is trivial, and if $\operatorname{len}(\mathcal{E})$ is a limit ordinal then it follows immediately from the induction hypothesis on $\operatorname{len}(\mathcal{E})$, together with the fact that $q' \le q$ implies that $\left\{ \iota < \eta : \nu^{q'}_\iota \ne \nu^q_\iota \right\}$ is bounded in η. Thus we can assume that $\operatorname{len}(\mathcal{E}) = \eta+1$, a successor ordinal. Furthermore we can assume that $q' \le q$ follows by a single application of one of the five clauses of definition 3.17. Again, the conclusion is trivial for the first clause, which has $\vec{\nu}^{q'} = \vec{\nu}^q$, and follows from the induction hypothesis on $\operatorname{len}(\mathcal{E})$ for the second clause.

If $q' \le q$ follows from the third clause, then condition 3.17(3b) implies that $\vec{C}_\nu$ is smoothly joined to $\vec{C}$ at $\vec{\nu}^q{\downarrow}\nu$, and condition 3.17(3a) together with definition 3.11(4b) implies that $\vec{C}_{f_\eta(\nu)}$ is smoothly joined to $\vec{C}$ at $\operatorname{add}(\vec{\nu}^q, \eta, \nu){\downarrow}f_\eta(\nu)$. Hence $\vec{\nu}^{q'} = \operatorname{add}(\vec{\nu}^q, \eta, f_\eta(\nu)) \prec \nu^q$ in P_λ.

Similarly, the induction hypothesis on λ, together with either condition 3.17(4c) or condition 3.17(5a), implies that $\vec{\nu}^{q'} = \operatorname{add}(\vec{\nu}^q, \eta, \nu) \prec \nu^q$ whenever $q' \le q$ follows from clause 4 or 5. $\square$

[[In fact I may as well move definition 3.22 and the proposition following to this point.]]

[[6/28/02 — I'm leaving this paragraph in the file at this point, since it more or less explains something which has had me confused some of the time. — — — We will later see (lemma 3.25) that in this case, whenever $\vec{C}$ is generic for $Q^{\mathcal{E}}_\lambda$ then C_η is generic for $Q^{\mathcal{E}\restriction\eta+1}_\lambda$, and if $\nu_0 = \min C_{\lambda,\eta}$ then for all $\iota > \eta$ we have $C_{\lambda,\iota} \setminus \nu_0 \subset C_{\lambda,\eta}$. For each member ν of $C_{\lambda,\iota} \setminus \nu_0$ we will have $o^*(\nu) > \iota{\downarrow}\nu > \eta{\downarrow}\nu$. Such an ordinal ν can be added to $C_{\lambda,\eta}$ in only two ways: via clause 4, or via the function f^q_η. Furthermore these routes are equivalent in the sense that one cannot specify, given a $Q^{\mathcal{E}\restriction\eta+1}_\lambda$ generic set $G_{\mathcal{E}}$ and ordinal ν, one of these alternatives as the one which for which ν was added to $C_{\lambda,\eta}$.]]

Corollary 3.19. *If $G_\lambda \subseteq Q^{\mathcal{E}}_\lambda$ is generic then the set $F_\lambda = \{ \vec{\nu} : \exists q \in G_\lambda\, \vec{\nu}^q \preccurlyeq \vec{\nu} \}$ is a filter on $P^{\vec{C}}_\lambda$.* $\square$

Lemma 3.20. *The set $\vec{D} = \operatorname{ext}^{\vec{C}}_\lambda(F_\lambda)$ is unbounded in λ if and only if $\gamma < o^*(\mathcal{E})$.*

Hence $o^*(\vec{D}) = o^*(\mathcal{E})$, and the sequence of length $\lambda+1$ obtained by settting $\vec{C}_\lambda = \vec{D}$ is a layered tree sequence.

Proof. If $\gamma \ge o^*(\mathcal{E})$ then $q' \le q$ implies $\nu^{q'}_\gamma = \nu^q_\gamma$, so $\max(D_\gamma) = \nu^q_\gamma$ for any $q \in G_\lambda$. Thus $o^*(\vec{C}) \le o^*(\mathcal{E})$.

It remains to show that $\gamma < o^*(\mathcal{E})$. We assume that $\mathcal{E}$ is semicomplete, so that $o^*(\mathcal{E}) = \operatorname{len}(\mathcal{E})$, and show by induction on $\operatorname{len}(\mathcal{E})$ that D_γ is unbounded in λ for all $\gamma < \operatorname{len}(\mathcal{E})$.

If $\gamma+1 < \operatorname{len}(\mathcal{E})$ then for any $\eta < \lambda$ and any condition $q \in Q^{\mathcal{E}}_\lambda$ there is, by the induction hypothesis, a $q' \le q|\gamma+1$ in $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$ such that $\nu^{q'}_\gamma > \eta$. Then $q'' = q'\restriction(\gamma+1)^\frown q{\lfloor}\gamma+1$ is a condition in $Q^{\mathcal{E}}_\lambda$ such that $q'' \le q$ and $\nu^{q''}_\gamma = \nu^{q'}_\gamma > \eta$.

It follows that it is forced in $Q^{\mathcal{E}}_\lambda$ that $C_{\lambda,\gamma}$ is unbounded in λ. This completes the proof of the lemma in the case $\operatorname{len}(\mathcal{E})$ is a limit ordinal, so for the rest of the proof we can assume that $\operatorname{len}(\mathcal{E}) = \gamma + 1$.

We can assume that $\beta^q_\gamma > 0$, since if $\beta^q_\gamma = 0$ then we can instead work with the condition $q' \leq^* q$ which is defined by setting $q' \restriction \gamma = q \restriction \gamma$ and $q'_\gamma = (\nu^q_\gamma, 1, A_\gamma, g_\gamma, \mathrm{id} \restriction A_\gamma)$ where

$$g_\gamma(\nu) = h_\gamma(\nu) \text{ and } A_\gamma = \left\{ \nu < \lambda : \bar{o}^{\mathcal{E}}(\nu) = 0 \ \ \& \ \ q{\downarrow}\nu \in Q^{\mathcal{E}\downarrow\nu}_\nu \right\}.$$

Then $A^{q'}_\gamma \in U(\lambda, \gamma \cdot \lambda^+) = \bar{U}^{\mathcal{E}}(\lambda, 0)$.

To complete the proof, fix any $\xi < \lambda$. Since $H_\lambda \subseteq \mathcal{R}_\lambda$ is generic, we can pick $\nu \in A^q_\gamma \setminus \xi$ such that $\bar{o}^{\mathcal{E}}(\nu) = 0$, $g^q_\gamma(\nu) \restriction \lambda \in H_\lambda$ and $(\mathcal{E}{\downarrow}\nu, (q|\gamma){\downarrow}\nu) \in G_\nu$. Then $\operatorname{add}^{\mathcal{E}}(q, \eta, \nu) \leq q$ since $\bar{o}^{\mathcal{E}}(\nu) = 0$ implies that $q|\gamma$ is the condition q'' of Definition 3.17(3a).

This completes the proof that $o^*(\vec{D}) = o^*(\mathcal{E})$. Now if D_0 is bounded then $\operatorname{len}(\mathcal{E}) = 0$ and hence F_λ has the minimal member $\vec{\nu}^q$, where q is the unique member of G_λ. Since $\vec{C} \restriction \lambda$ is a layered tree sequence, it follows that the sequence of length $\lambda + 1$ obtained by setting $\vec{C}_\lambda = \vec{D}$ is a layered tree sequence. This completes the proof of lemma 3.20. □

Proposition 3.21. *Any condition $q \in Q^{\mathcal{E}}_\lambda$ forces that $C_{\lambda,\gamma} \setminus \nu^q_\gamma \subset \operatorname{val}^{H_\lambda}(E_\eta, h_\eta)$ for all $\eta < o^*(\lambda)$.*

Proof. The proof is by induction on $\operatorname{len}(\mathcal{E})$, and as in previous arguments we need consider only the case $\operatorname{len}(\mathcal{E}) = \gamma + 1$. We will show that if q' and $q'' = \operatorname{add}^{\mathcal{E}}(q', \gamma, \nu)$ are in $G^{\mathcal{E}}_\lambda$ then $C_{\lambda,\gamma} \cap (\nu^{q'}, \nu^{q''}] \subset \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$. Since $\lambda \setminus \nu^q_\gamma$ is the union of such intervals this will prove the proposition.

We must have $q'' \leq q'$ by one of the clauses 3-5 of Definition 3.17, since the first two clauses do not increase $\nu^{q'}_\gamma$. For each of these clauses, subclause (a) implies that $C_{\lambda,\gamma} \cap (\nu^{q'}_\gamma, \nu) \subset \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$.

In the case that $q'' \leq q'$ by clause 4 or 5, subclause (b) together with the fact that $q'' \| \gamma$ is $h_\eta(\nu)$ implies that $\nu = \nu^{q''}_\gamma \in \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$, and this completes the proof for this case.

In the case that $q'' \leq q'$ by clause 3 we need to look at clause 3 of the definition 3.11 of the set $Q^{\mathcal{E}}_\lambda$ of conditions. If $f^{q'}_\gamma(\nu) > \nu$ then subclause (c) implies that $C_{\lambda,\gamma} \cap [\nu, f^{q'}_\gamma) \subset \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$. Finally subclause (d) implies that $\nu^{q''} = f^{q'}_\gamma(\nu) \in \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$, and this completes the proof that $C_{\lambda,\gamma} \cap (\nu^{q'}_\gamma, \nu^{q''}_\gamma] \subset \operatorname{val}^{H_\lambda}(E_\gamma, h_\gamma)$, and this completes the proof of the proposition. □

This next is awkward, and also is probably redundant.

Next we look at the case of a complete sequence $\mathcal{E}$ of successor length, and describe a dense subset of $Q^{\mathcal{E}}_\lambda$ in a way which may be more reminiscent of Gitik's forcing in [1]. In order to see the resemblance one should identify $\nu \in P^{\mathcal{E}}(q)$ with the closed set $C_{\nu,\gamma\downarrow\nu} \setminus \nu^q_\gamma$ and note that our forcing differs from Gitik's in that where his forcing allows arbitrary bounded closed sets as conditions, our forcing is restricted to the tree of sets of this form.

Definition 3.22. If $\vec{\nu} \in P_\lambda$ then the partial order $(P^{\mathcal{E}}(\vec{\nu}), \preceq)$ is defined by setting $P^{\mathcal{E}}(\vec{\nu})$ equal to the set of ordinals ν with $\nu_\eta \leq \nu < \lambda$ such that $t^{\mathcal{E}}_\nu(\vec{\nu})$ is defined, and writing $\nu' \preceq \nu$ if and only if $t^{\mathcal{E}}_{\nu'}(\vec{\nu}) \leq t^{\mathcal{E}}_\nu(\vec{\nu})$.

If $q \in Q^{\mathcal{E}}_\lambda$ then we will write $P^{\mathcal{E}}(q)$ for $P^{\mathcal{E}}(\vec{\nu}^q)$ and $t^{\mathcal{E}}_\nu(q)$ for $t^{\mathcal{E}}_\nu(\vec{\nu}^q)$.

Proposition 3.23. *Suppose* $\operatorname{len}(\mathcal{E}) = \eta + 1$*,* $\mathcal{E}$ *is complete, and* $q \in Q^{\mathcal{E}}_\lambda$*. Then the map* $\nu \mapsto t^{\mathcal{E}}_\nu(q)$ *embeds* $P^{\mathcal{E}}(q)$ *onto a* $\leq$*-dense subset of* $\{\, q' \in Q^{\mathcal{E}}_\lambda : q' \leq q \,\}$*.*

Proof. The map $\nu \mapsto t^{\mathcal{E}}_\nu(q)$ is an order preserving embedding by the definition of $(P^{\mathcal{E}}(q), \preccurlyeq)$. To see that the range of the embedding is dense, suppose that $q' \leq q$. Set $\beta = \beta^{q'}_\eta < \lambda^+$. Then $\{\, \nu' : \bar{o}^{\mathcal{E}}(\nu') = \beta \,\} \in \overline{U}^{\mathcal{E}}(\lambda, \beta)$. Thus there will certainly be some ν' with $o^{\mathcal{E}}(\nu') = \beta$ such that $\vec{\nu}^{q'} \propto_{\nu'} \eta$ and $(\mathcal{E}, q')\!\downarrow\!\nu' \in G'_\nu$. Then $t^{\mathcal{E}}_{\nu'}(q) \leq q'$ by definition 3.17(4). $\square$

Note that the assumption that $\mathcal{E}$ is complete is needed here. If $\mathcal{E}$ is not complete then $\left\{\, q' : \beta^{q'}_\eta = \bar{o}^{\mathcal{E}}(\lambda) \,\right\}$ is dense. If q' is any member of this set then only clauses (1–3) of definition 3.17 can apply, and these clauses cannot give $t^{\mathcal{E}}_{\nu'}(q) \leq q'$.

Corollary 3.24. *Suppose that* $\mathcal{E}$ *is complete sequence on* λ *with* $\operatorname{len}(\mathcal{E}) = \eta + 1$*, and* $q \in G_\lambda$*. Then* $C_{\lambda,\eta} \setminus \nu^q_\eta$ *is a* $V[H_\lambda]$*-generic subset of* $P^{\mathcal{E}}(q)$*.* $\square$

The following important lemma says that the forcing $Q^{\mathcal{E}}_\lambda$ can be factored as $Q^{\mathcal{E}}_\lambda \equiv Q^{\mathcal{E}\restriction\gamma+1}_\lambda * \dot{S}^{\mathcal{E}}_{\lambda,\gamma}$ for some forcing $S^{\mathcal{E}}_{\lambda,\gamma}$. This fact will be used in section 4, where we will need to look more carefully at the second factor $S^{\mathcal{E}}_{\lambda,\gamma}$.

Note that it is not that $G_\lambda \cap Q^{\mathcal{E}\restriction\gamma+1}_\lambda$ is generic. The generic subset of $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$ is obtained (given some $\vec{\nu}$ coming from an member of G_λ—preferably the $P^{\vec{C}}_\lambda$-maximal one) map a condition $q \in G_\lambda$ to the sequence with sequence $\vec{\nu}^q\restriction\gamma+1^\frown\vec{\nu}\lfloor\gamma+1$.

Note: this is a new thought, which would require some checking and revision. (No—it doesn't work because this sequence isn't decreasing on its support.)

6/27/07 — See the comment following proposition 1.12.

Lemma 3.25. *Suppose that* $\gamma+1 < \operatorname{len}(\mathcal{E})$*, that* $\mathcal{E}\restriction\gamma+1$ *is complete, and that* $q \in G_\lambda$ *where* G_λ *is a* $V[H_\lambda]$*-generic subset of* $Q^{\mathcal{E}}_\lambda$*. Then the set* $C_{\lambda,\gamma} \setminus \nu^q_\gamma$ *is a generic subset of* $P^{\mathcal{E}\restriction\gamma+1}(\vec{\nu}^q)$*, and hence* $G' = \{\, (q'\restriction\gamma+1)^\frown(q|\gamma+1)\lfloor\gamma+1 : q' \in G_\lambda \,\}$ *is a generic subset of* $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$*.*

Proof. Since $P^{\mathcal{E}\restriction\gamma+1}(q)$ is embedded onto a dense subset of $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$, the assertion about G' follows from that about $C_{\lambda,\gamma} \setminus \nu^q_\gamma$. Since the lemma is trivial if $\operatorname{len}(\mathcal{E}) = \gamma + 1$, we can assume $\gamma + 1 < \operatorname{len}(\mathcal{E})$.

Let q be an arbitrary condition in $Q^{\mathcal{E}}_\lambda$, Let $\dot{D}$ be a $\mathcal{R}_\lambda$ name for a dense subset $D \subset P^{\mathcal{E}\restriction\gamma+1}_\lambda$, and let $p \in \mathcal{R}_\lambda$ force that $\dot{D}$ is dense.

Set $U = \mathcal{U}^{\mathcal{E}}(\lambda, (\gamma+1)\cdot\lambda^+)$, which exists since $\operatorname{len}(\mathcal{E}) > \gamma + 1$. There is a set $A \in U$ such that for $\nu \in A$ we have that $\bar{\mathcal{E}} = (\mathcal{E}\restriction\gamma+1)$ and $q\!\downarrow\!\nu$ exist, and p forces that $\dot{D}$ is a dense subset of $P^{\bar{\mathcal{E}}}(q\!\downarrow\!\nu)$.

We can define $p' \le p$ in $\mathcal{R}_\lambda$ and q' in $Q^{\mathcal{E}}_\lambda$ and $\nu \in A$ so that $p' \Vdash q' \le q$, $p'_\nu = (\bar{\mathcal{E}}, (q|\gamma+1){\downarrow}\nu)$ and $\nu^{q'}_{\gamma+1} = \nu$. [[Also $p' \Vdash$ "$\dot{D} \cap H_\lambda$ is a $\mathcal{R}_\lambda$-name for $\dot{D} \cap \nu$".]]

Then since $C_{\nu,\gamma\downarrow\nu}$ is a generic subset of $P^{\bar{\mathcal{E}}}(q{\downarrow}\nu)$, there is some $\nu' \in C_{\nu,\gamma\downarrow\nu} \subset C_{\lambda,\gamma}$ which is in D. □

7/20/07 — MUCH Work needed in this area.

Proposition 3.26. *If $\mathcal{E}$ is complete then $Q^{\mathcal{E}}_\lambda$ extends $\vec{C}(H_\lambda)$.*

Proof. Let $G_\lambda \subseteq Q^{\mathcal{E}}_\lambda$ be generic and define f by setting $f(q) = \vec{\nu}^q$. First suppose that $\mathrm{len}(\mathcal{E}) = \eta + 1$. By lemma 3.20 $f``G_\lambda$ generates a filter F_λ on $P^{\vec{C}}_\lambda$ which extends $\vec{C}{\restriction}\lambda$ to a layered tree sequence of length $\lambda+1$. By Corollary 3.24, the set $G^{\mathcal{E}}_\lambda$ is determined by the set $C_{\lambda,\eta}$. Thus the function f witnesses that $Q^{\mathcal{E}}_\lambda$ extends $\vec{C}{\restriction}\lambda$.

Now suppose that $\eta = \mathrm{len}(\mathcal{E})$ is a limit ordinal. That the filter generated by $f``G_\nu$ extends the layered tree sequence $\vec{C}{\restriction}\lambda$ follows from Lemmas 3.25 and 3.27.

> 8/15/07 *** The other part, that G_λ can be recovered from $\vec{C}_\lambda$, doesn't seem to me to be obvious. This will probably work: $q \in G_\lambda$ if an only if $q|\xi+1$ is in the generic subset of $Q^{\mathcal{E}\restriction\xi+1}_\lambda$ induced by $\vec{C}_\lambda{\restriction}\xi$ for each $\xi < \gamma$. But it is not obvious that this should work.
>
> This might have to go with the incomplete sequences.??????

□

The proof that $Q^{\mathcal{E}}_\lambda$ extends $\vec{C}(H_\lambda)$ for an incomplete sequence $\mathcal{E}$ is more difficult, and will be deferred to the end of this section, after the proof of the Prikry property, at which time we will also show that Q_λ extends $\vec{C}(H_\lambda)$, that is, that the sequence $\mathcal{E}$ can be recovered (up to the equivalence described in Definition 3.1) from $\vec{C}(G_\lambda)$.

> I also need to say something about the case that $\mathrm{len}(\mathcal{E})$ is a limit ordinal.
>
> This should work for the limit case: For any $q \in Q^{\mathcal{E}}_\lambda$ and let $\bar{Q}^{\mathcal{E}}(q)$ be the set of conditions $q' \le q$ in $Q^{\mathcal{E}}_\lambda$ such that for some $\eta < \mathrm{len}(\mathcal{E})$ and $\nu < \lambda$ we have $q'|\eta = t^{\mathcal{E}}_\nu(q)$ and $q' \le^* t^{\mathcal{E}}_\nu(q)$. Then $\bar{Q}$ is dense in $Q^{\mathcal{E}}_\lambda/q$.
>
> Then (given $\mathcal{E}$) G_λ would be the set of q such that for arbitrarily large ν there is some $q' \le t^{\mathcal{E}}_\nu(q)$ such that $q' \in \bar{Q}^{\mathcal{E}}(q)$.

[[The following is in preparation for the Prikry property(?)]]

Proposition 3.27. *The ordering $(Q^{\mathcal{E}}_\lambda, \le^*)$ is $<\lambda$-closed.*

Proof. Suppose that $\delta < \lambda$ and $\langle q^\xi : \xi < \delta \rangle$ is a $\le^*$-decreasing sequence. To define q, set $\vec{\nu}^q = \vec{\nu}^{q^0}$ and $\beta^q_\gamma = \sup\left\{ \beta^{q^\xi}_\gamma : \xi < \delta \right\}$. Define $A^q_\gamma = \liminf\left\{ A^q_{\gamma^\xi} \setminus \delta : \xi < \delta \right\} = \left\{ \nu < \lambda : \exists \xi < \delta \forall \xi' > \xi \; \nu \in A^{q^\xi}_\gamma \setminus \delta \right\}$. Thus if $\nu \in A^q_\gamma$ then $\nu \in A^{q^\xi}_\gamma$ for all $\xi < \delta$ such that $\bar{o}^{\mathcal{E}}(\nu) < \beta^{q^\xi}_\gamma$.

Finally, set $f^q_\gamma(\nu) = f^{q^\xi}_\gamma(\nu)$ for all $\xi < \delta$ such that $\nu \in A^{q^\xi}_\gamma$, and set $g^q_\gamma(\nu) = \bigwedge\left\{ g^{q^\xi}_\gamma(\nu) : \nu \in A^{q^\xi}_\gamma \right\}$.

> 8/15/07 — This is an argument by induction; I need that $Q^{\mathcal{E}\restriction\gamma}_\lambda$ is $< \lambda$-closed to get that $\bigwedge\left\{ g^{q^\xi}_\gamma(\nu) : \nu \in A^{q^\xi}_\gamma \right\}$ exists.

□

> This definition of A^q_γ is equivalent to saying $\nu \in A^q_\gamma$ if and only if $\bar{o}^{\mathcal{E}}(\nu) < \beta^q_\gamma$ and $\nu \in A^{q^\xi}_\gamma \setminus \delta$ for all $\xi < \delta$ such that $\bar{o}^{\mathcal{E}}(\nu) < \beta^{q^\xi}_\gamma$.

Definition 3.28. If $q \in Q^{\mathcal{E}}_\lambda$ and $\nu < \lambda$ then q/ν is the condition defined by $(q/\nu)\iota = \langle \nu^q_\iota, \beta^q_\iota, A^q_\iota \setminus \nu, g^q_\iota \restriction A^q_\iota \setminus \nu, f^q_\iota \restriction A^q_\iota \setminus \nu \rangle$ for each $\iota < \lambda^+$.

A sequence $\langle q^\xi : \xi < \eta \rangle$ is *diagonally decreasing* if $\beta^{q^\xi}_\iota > 0$ for all $\xi < \eta$ and $\iota < \bar{o}^{\mathcal{E}}(\lambda)$. and $q^{\xi'}/\nu \leq^* q^\xi$ whenever $\xi' \propto_\nu \xi$.

Note that $\iota < \lambda^+$ and $q \in Q^{\mathcal{E}}_\lambda$ satisfies β^q_ι then $q/\nu \Vdash C_{\lambda,\iota} \cap \nu \subseteq \nu^q_\iota + 1$.

Proposition 3.29. *If $\langle q^\xi : \xi < \eta \rangle$ is a diagonally decreasing sequence then there is $q = q^\eta = \Delta_{\xi<\eta} q^\xi$ such that $\langle q^\xi : \xi < \eta + 1 \rangle$ is diagonally descreasing.*

Proof. Set $\vec{\nu}^q = \vec{\nu}^{q^0}$ and $\beta^q_\iota = \sup_{\xi<\eta} \beta^{q^\xi}_\iota$.

Let A^q_ι be the set of $\nu < \lambda$ such that $\bar{o}^{\mathcal{E}\restriction\iota+1}(\nu) < \beta^q_\iota$ and $\nu \in A^{q^\xi}_\iota$ for all $\xi \propto_\nu \eta$ such that $\bar{o}^{\mathcal{E}\restriction\iota+1}(\nu) < \beta^{q^\xi}_\iota$. Set $g^q_\iota(\nu) = \bigwedge\left\{ g^{q^\xi}(\nu) : \xi \propto_\nu \eta \;\&\; \nu \in A^{q^\xi}_\iota \right\}$, and let $f^q_\iota(\nu) = f^{q^\xi}_\iota(\nu)$ for any $\xi \propto_\nu \eta$ such that $\nu \in A^{q^\xi}_\iota$.

== The only thing to check seems to be that $\langle q^\xi/\nu : \xi \propto_\nu \eta \rangle$ is $\leq^*$-decreasing. This follows from the properties of the sets $A_{\xi,\nu}$ from [8]: If $\xi' \in A_{\xi,\nu}$ then $A_{\xi',\nu} = A_{\xi,\nu} \cap \xi'$ so $\xi'', \xi' \propto_\nu \xi$ implies $\xi'' \propto_\nu \xi'$. □

> "Pure" is not a very good name and includes two notions which are not really related. But is it worth the trouble to change this?
>
> I probably want to eliminate at least condition (i).

Definition 3.30. We say that q is *pure* if (i) $\gamma \propto_\nu \operatorname{len}(\mathcal{E})$ for each $\gamma < \operatorname{len}(\mathcal{E})$ and $\nu \in A^q_\gamma$, and (ii) if $\operatorname{len}(\mathcal{E}) = \eta + 1$ and $\mathcal{E}$ is not complete then $\beta^q_\eta = \bar{o}^{\mathcal{E}}(\lambda)$ and $\bar{o}^{\mathcal{E}}(\nu)$ is defined for all $\nu \in A^q_\eta$.

The pure conditions form a dense, open subset of $(Q^{\mathcal{E}}_\lambda, \leq^*)$. NEEDS WORK

> The first clause is useful because it provides a uniform way to define diagonal limits in $Q^{\mathcal{E}}_\lambda$ of length $\eta \leq \operatorname{len}(\mathcal{E})$:
>
> I haven't said anything about β^q in this — which is a problem because the sets $A^{q^\xi}_\nu$ are not decreasing — even for short limits — because the ordinals β^q_γ could be increasing.

Proposition 3.31. *Suppose that q^0 is pure and $q = \Delta_\xi q^\xi$. Then $q/\nu \leq^* q^\xi$ for each $\xi < \operatorname{len}(\mathcal{E})$ and $\nu < \lambda$ such that $\xi \propto_\nu \operatorname{len}(\mathcal{E})$.*

In particular, for all such ξ and ν we have $\operatorname{add}(q,\gamma,\nu) \le^* \operatorname{add}(q^\xi,\gamma,\nu)$. *Furthermore, if $\nu' < \nu$ and p satisfy $\gamma \propto_{\nu'} \operatorname{len}(\mathcal{E})$ and $p \Vdash \operatorname{add}(q,\gamma,\nu) \le q$, then $p' \Vdash \operatorname{add}(q,\gamma,\nu) \le^* \operatorname{add}(q^\xi,\gamma,\nu) \le q^\xi$, where p' is the same as p except that $p'_\nu = p_\nu/\nu'$.* □

Proof. In the second paragraph, the point is that any $q' \le \operatorname{add}(q,\gamma,\nu)$ has $\nu_\iota^{q'} > \nu$ whenever $\nu_\iota^{q'} > \nu_\iota^q$ and $\nu_\iota^{\operatorname{add}(q,\gamma,\nu)} \ne \nu$.

□

To see why the second clause of the definition of a pure condition is useful, we recall definition 2.5 from [7]:

The emphasis here should be on the equivlence with Prikry-Magidor forcing, not with this notion of pure. Put the second clause of the current "pure" in the hypothesis of this (or a lemma depending on it). Note that this set is *open* and $\le^*$-dense.

In the introduction: state refering to $\le$ unless otherwise stated. Thus open and $\le^*$-dense means open in $\le$ — ie, once in the set you never get out of it.

Definition 3.32. $\overline{C}_{\lambda,\eta,\nu,f}$ is the closed subset of $C_{\lambda,\eta}$ whose initial and successor members are defined as follows:

$$\min\left(\overline{C}_{\lambda,\eta,\nu,f}\right) = \min\left(C_{\lambda,\eta} \setminus \nu+1\right)$$
$$\forall \nu' \in \overline{C}_{\lambda,\eta,\nu,f}\ \min\left(\overline{C}_{\lambda,\eta,\nu,f} \setminus (\nu'+1)\right) = \min\left(C_{\lambda,\eta} \setminus (f(\nu')+1)\right).$$

Any condition q with $\nu = \nu_\eta^q$ and $f = f_\eta^q$ forces that $\overline{C}_{\lambda,\eta,\nu,f}$ is unbounded in λ. If q is pure then q forces that $\overline{C}_{\lambda,\eta,\nu,f}$ is a pure Prikry-Magidor set. If $\bar{o}^{\mathcal{E}}(\lambda) < \lambda$ then this implies that $\operatorname{otp}(\overline{C}_{\lambda,\eta,\nu,f}) = \omega^{\bar{o}^{\mathcal{E}}(\lambda)}$.

The point being that thus the forcing for incomplete $\mathcal{E}$ of successor length is equivalent to Prikry-Magidor forcing. The assumption that $\beta_\eta^q = \bar{o}^{\mathcal{E}}(\lambda)$ says that q is beyond the point at which the forcing can behave like forcing for complete $\mathcal{E}$ (before which almost any extension in $P_\lambda^{\vec{C}}$ could be made) and the statement that $\bar{o}^{\mathcal{E}}(\nu)$ exists for all $\nu \in A_\eta^q$ says that there are no oddball extensions from A_η^q.

3.3 Prikry Property (New 11/15/07)

11/16/07 — This section is intended to replace all or most of the existing sections 3.4, 3.5, and 3.6.

I have tried to outline the intended proof in the OmniOutliner file

The main result of this section is that $Q_\lambda^{\mathcal{E}}$ satsifies the Prikry Property:

Lemma 3.33. *Suppose that $\mathcal{E}$ is a suitable sequence on λ. Then for every sentence σ and for every $q \in Q_\lambda^{\mathcal{E}}$ there is $q' \le^* q$ such that $\Vdash_{\mathcal{R}_\lambda} q' \parallel_{Q_\lambda^{\mathcal{E}}} \sigma$.*

Most of the work towards the proof of lemma 3.33 will go towards proving lemma 3.35 below, which relies on the following definition:

Definition 3.34. An extension $q' \leq^* q$ is said to be γ-bounded if $\nu^{q'}_{\gamma'} = \nu^{q}_{\gamma'}$ for all $\gamma' > \gamma$, and the extension $q' \leq q$ is *strongly γ-bounded* if $q'_{\gamma'} = q'_{\gamma}$ for all $\gamma' > \gamma$.

Lemma 3.35. *Suppose $\gamma + 1 < \operatorname{len}(\mathcal{E})$, $\xi < \lambda$, σ is a sentence, and $q \in Q^{\mathcal{E}}_{\lambda}$. Then there is a $\leq^*$-dense set of conditions $q' \leq^* q$ such that the following property is forced in $\mathcal{R}_\lambda$:*

Let $q'' \leq^* q$ be any condition such that $q''/\xi \leq^* q'$ and $A^{q''}_{\gamma} \cap \xi + 1 = \emptyset$, and suppose that there is a $\gamma+1$-bounded extension of q'' which decides σ. Then there is a γ-bounded extension of q'' which decides σ. [[I'm not sure that I need to require $q'' \leq^* q$. If not, then I shouldn't have to mention q at all — in which case I could drop a prime from everything.]]

Proof of lemma 3.33 from lemma 3.35. Let q be arbitrary, and define a $\leq^*$-descending sequence $\langle q^{\xi} : \xi < \lambda \rangle$ so that if $\xi < \lambda$ and $\gamma \propto_{\gamma} \eta$ then q^{ξ} satisfies the conclusion of Lemma 3.35 for γ and ξ. Now define $q'' \leq^* \Delta_{\xi<\lambda} q^{\xi}$ by setting $A^{q''}_{\gamma} \cap \xi_{\gamma} + 1 = \emptyset$ for each $\gamma < \eta$, where ξ_γ is least such that $\gamma \propto_{\xi} \eta$. Thus q'' satsifies the hypothesis of lemma 3.35 with respect to q^{ξ} for each $\gamma \propto_{\xi} \eta$.

I claim that there is $q''' \leq^* q''$ which decides σ. If this is not so, then there is $\gamma + 1 \leq \operatorname{len}(\mathcal{E})$ such that there is a $\gamma+1$-bounded extension of q'' deciding σ, but no γ-bounded extension, and this contradicts the choice of q^{ξ}. □

Definition 3.36. If σ is a sentence of set theory then $\mathcal{X}^{\sigma}_0$ is the set of conditions q such for each $\gamma < \operatorname{len}(\mathcal{E})$ and $\gamma+1$-bounded extension $q' \leq q$ there is a strongly $\gamma+1$-bounded $q'' \leq q$ such that $q' \leq q''$ and q'' decides σ.

Lemma 3.37. *For any sentence σ, the set $\mathcal{X}^{\sigma}_0$ is dense in $(Q^{\mathcal{E}}_{\lambda}, \leq^*)$.*

Proof. Let $q^0 \in Q^{\mathcal{E}}_{\lambda}$ be arbitrary; we will find a condition $q \leq^* q^0$ in $\mathcal{X}^{\sigma}_0$. Since any condition q which decides σ is in $\mathcal{X}^{\sigma}_0$, we can assume for the rest of the proof that no $q \leq^* q^0$ decides σ.

We will define a $\leq^*$-decreasing sequence $\langle q^{\iota} : \iota < \lambda \rangle$ below q such that each condition q^{ι} satisfies the following property:

$$\forall q \leq^* q^{\iota}\, \forall \gamma \propto_{\iota} \eta \forall \nu < \lambda \ \left(q_{*\gamma,\nu} \parallel \sigma \implies q^{\iota}_{*\gamma,\nu} \parallel \sigma\right). \tag{3}$$

Because $Q^{\mathcal{E}}_{\lambda}$ is $\leq\lambda$-closed, in order to define $q^{\iota} \leq^* \bigwedge_{\iota'<\iota} q^{\iota'}$ satisfying (3) it will be sufficient to show that for each condition q and each $\gamma < \operatorname{len}(\mathcal{E})$ there is $q' \leq^* q$ satisfying (3) for that choice of γ. To do this, we define a $\leq^*$-decreasing sequence $\langle r^{\nu} : \nu < \lambda \rangle$ below q such that for each $\nu < \lambda$, $r^{\nu} \restriction \gamma + 1 = q \restriction \gamma + 1$, and either $r^{\nu}_{*\gamma,\nu} \parallel \sigma$, or else no $r \leq^* r^{\nu}_{*\gamma,\nu}$ with $r \restriction \gamma + 1 = r^{\nu} \restriction \gamma + 1$ decides σ.

Now define $q' = \Delta_{\nu<\lambda} r^{\nu}$. To see that q' satisfies (3) for this γ, suppose that $q'' \leq^* q'$ and $q''_{*\gamma,\nu} \parallel \sigma$. Then $q''_{*\gamma,\nu} = (q''/\nu)_{*\gamma,\nu}$, and since $q''/\nu \leq^* r^{\nu}$ it follows that $r^{\nu}_{*\gamma,\nu} \parallel \sigma$. But $q'_{*\gamma,\nu} = (q'/\nu)_{*\gamma,\nu} \leq^* r^{\nu}_{*\gamma,\nu}$, so (3) holds for γ.

I claim that the condition $q = \Delta_{\iota<\lambda} q^{\iota}$ is in $\mathcal{X}^{\sigma}_0$. To see this, suppose that $q' \leq q$ decides σ, and let γ be the largest ordinal such that $\nu^{q'}_{\gamma} > \nu^{q}_{\gamma}$. Then $q' \leq q$ is $\gamma+1$-bounded, but not γ-bounded. Set $q'' = (q' \restriction \gamma + 1)^\frown (q \lfloor \gamma + 1)$. Then $q' \leq q'' \leq q$ and $q'' \leq q$ is strongly $\gamma+1$-bounded, so it will be sufficient to

verify that $q'' \parallel \sigma$. For this, in turn, it will be sufficient to show that q'' forces that there is $\nu < \lambda$ such that $q''_{*\gamma,\nu} \in \dot{G}_\lambda$, for then $q'_{*\gamma,\nu} \leq q'$, so that $q'_{*\gamma,\nu} \parallel \sigma$, but since $q'\lfloor\gamma,\nu \leq^* q\lfloor\gamma,\nu$ it then follows by (3) that $q''_{*\gamma,\nu} \parallel \gamma$.

Now suppose that $q''' \leq q''$. If this extension is $\gamma+1$-bounded then it is clear that there is ν as desired, so we can assume that $\nu^{q'''}_{\gamma'} > \nu^{q}_{\gamma'}$ for some $\gamma' > \gamma$. We can suppose that for some ν', q''' forces that ν' is the least member of $\bigcup\left\{ C_{\lambda,\gamma'} \setminus \nu^{q}_{\gamma'} : \gamma < \gamma' < \operatorname{len}(\mathcal{E}) \right\}$. A little thought, using the fact that $q'' \leq q$ is $\gamma+1$-bounded, but not γ-bounded, shows that $q'' \leq \operatorname{add}(q'', \gamma+1, \nu) \leq q'''$. However $\operatorname{add}(q'', \gamma+1, \nu)$ forces that there is $\nu'' < \nu'$ so that $\operatorname{add}(q'', \gamma+1, \nu) \leq q''_{*\gamma,\nu''} \leq q''$, and it follows, as before, that $\operatorname{add}(q'', \gamma+1, \nu)$ decides σ. [1] □

We break up the proof of Lemma 3.35 into three cases. The first case, $\gamma+1 = \operatorname{len}(\mathcal{E})$ and $\mathcal{E}$ is not complete, will be proved by adapting the usual proof of the Prikry property. The second case, $\gamma+1 = \operatorname{len}(\mathcal{E})$ and $\mathcal{E}$ is complete, will be reduced to the first case. The final case, $\gamma+1 < \operatorname{len}(\mathcal{E})$ will be reduced, using Lemma 3.37, to the second case applied to $\mathcal{E}\restriction\gamma+1$.

3.3.1 Case 2: $\operatorname{len}\mathcal{E} = \gamma+1$ and $\mathcal{E}$ is complete

We assume that q, σ, γ and ξ are as in the statemnet of the lemma. Set $\beta = \beta^q_\gamma$ and let $A = \left\{ \nu \in \lambda \setminus \xi+1 : \bar{o}^{\mathcal{E}\restriction\gamma}(\nu) = \beta \right\}$.

This is no good: $\bar{o}(\nu)$ is not defined, since $\mathcal{E}$ is complete. Try: $A \subseteq \lambda \setminus ((\xi+1) \setminus A^a_\gamma$ and $\nu \in A$ implies $\mathcal{E}{\downarrow}\nu$ exists and $\bar{o}^{\mathcal{E}\downarrow\nu}(\nu) = \beta{\downarrow}\nu$.

We will have $\beta^{q'}_\gamma = \beta+1$. [[Trying to save notation, let's keep using q. We first define $g_\gamma(\nu)$ for $\nu \in A$.

Claim 3.38. *We can define $f_\gamma\restriction A$ and $g_\gamma\restriction A$ so that for each $\nu \in A$,*

$$\Vdash_{\mathcal{R}_{\nu+1}} g_\gamma(\nu) \Vdash_{\mathcal{R}_{\nu+1,\lambda}} q_{*\gamma,f_\gamma(\nu)}) \parallel \sigma.$$

Proof. The proof of Claim 3.38 follows immediately from Lemma 3.39 below. □

I'm not properly using the established notation here. I'll have to go from the beginning and then rewrite this when I see what's going on. In particular the $q_{*\gamma,f_\gamma(\nu)}$ should be t-something.

Lemma 3.39 corresponds to the observation in the proof of theorem 1.1 of Gitik [1] which asserts that his forcing $P[E]$ is distributive. The hypothesis is, of course, implied by the assumption that $\mathcal{E}$ is complete.

Lemma 3.39. *Suppose that* $\operatorname{len}(\mathcal{E}) = \gamma+1$ *and* $\mathcal{E}\restriction\gamma$ *is complete, and that for each* $\zeta < \lambda$ *the set*

$$X_\zeta = \left\{ \nu < \lambda : \mathcal{E}{\downarrow}\nu \text{ is defined and semicomplete and } \bar{o}^{\mathcal{E}\downarrow\nu}(\nu) = \beta < \nu \right\}$$

[1] I think that I need to modify the difinitioin of q' by shrinking the sets $A^{q'}_{\gamma'}(0)$ so that the statement above follows by elementarity.

is stationary. Then $P^{\mathcal{E}}(\vec{\nu})$ is $<\lambda$-distributive.

Indeed, if $\{ D_\alpha : \alpha < \tau \}$ is a set of $\tau < \lambda$ open dense subsets of $P^{\mathcal{E}}(\vec{\nu})$ then for any $\nu \in P^{\mathcal{E}}(\vec{\nu})$ there is a cardinal $\xi > \nu$ and a condition $p \in \mathcal{R}_\lambda$ with domain$(p) = \{\xi\}$ *such that $p \Vdash_{P^{\mathcal{E}}_{\vec{\nu}}} (\xi \preccurlyeq \nu$ and $\xi \in \bigcap_{\alpha<\tau} D_\alpha)$.*

Proof. For $\alpha < \tau$, $\nu < \lambda$ and $p \in \mathcal{R}_\nu$ define $\mu(\alpha, \nu, p)$ to be the least ordinal $\mu > \nu$ such that $p' \Vdash_{\mathcal{R}_\lambda} \mu \in \dot{D}_\alpha$ for some $p' \leq p$ with $p' \restriction \nu + 1 = p$. If $\mu(\alpha, \nu, p)$ is defined, then set $\ell(\alpha, \nu, p) = p' \restriction \nu + 1$ for some such condition p'.

I claim that for each α there is a dense set Y_α of conditions $p \in \mathcal{R}_\lambda$ such that $\mu(\alpha, p, \nu)$ is defined for all $\nu < \lambda$ such that $p \in \mathcal{R}_\nu$. To see this, fix $\alpha < \tau$ and $p \in \mathcal{R}_\lambda$ and for each $\nu < \lambda$ choose $p_\nu \leq p$ in $\mathcal{R}_\lambda$ and $\xi_\nu > \nu$ so that $p_\nu \Vdash \xi_\nu \in \dot{D}_\alpha$. There is a stationary set S of ν such that $p_\nu \restriction \nu$ is constant, say $p_\nu \restriction \nu = p'$. Then $\mu(\alpha, p', \nu)$ is defined for all $\nu > \sup \mathrm{domain}(p') + 1$, since $p_{\nu'}$ witnesses that $\mu(\alpha, p', \nu) \leq \xi_{\nu'}$ for any $\nu' \in S \setminus \nu + 1$. Hence $p' \in Y_\alpha$.

The hypothesis to lemma 3.39 implies that λ is Mahlo, and hence $\mathcal{R}_\lambda$ has the λ-chain condition. It follows that there is $\delta < \lambda$ so that Y_α is dense in $\mathcal{R}_\delta$ for all $\alpha < \tau$. Let B be the set of $\xi < \lambda$ such that $\mu(\alpha, p, \nu) < \xi$ and $\ell(\alpha, p, \nu) \in \mathcal{R}_\xi$ for all $\alpha < \tau$, $p \in Y_\alpha \cap \mathcal{R}_\delta$ and $\nu < \xi$. Then B is closed and unbounded, so there is some $\xi > \delta$ in $B \cap X_\delta$. I will show that there is a condition $\bar{q} \in Q^{\bar{\mathcal{E}}}_\xi$ such that $\Vdash_{\mathcal{R}_\xi} \bar{q} \Vdash_{Q^{\bar{\mathcal{E}}}_\xi} \nu \in \bigcap_{\alpha<\tau} \dot{D}_\alpha$. For this it will be sufficient to arrange that $C_{\xi,\bar{\gamma}} \cap D_\alpha \neq \emptyset$.

To define $\bar{q}_{\bar{\gamma}}$, set $\beta^{\bar{q}}_{\bar{\gamma}} = \delta$ and $A^{\bar{q}}_{\bar{\gamma}} = \left\{ \nu \in \cap\delta \setminus \xi : \bar{o}^{\mathcal{E}\downarrow\xi} < \delta \right\}$. Now let $\langle (\alpha_\iota, p_\iota) : \iota < \delta \rangle$ enumerate the set of pairs (α, p) such that $\alpha < \tau$ and $p \in Y_\alpha \cap \mathcal{R}_\delta$. Then for $\nu \in A^{\bar{q}}_{\bar{\gamma}}$ we set $f^{\bar{q}}_{\bar{\gamma}}(\nu) = k(\alpha_{\bar{o}^{\bar{\mathcal{E}}}(\nu)}, p_{\bar{o}^{\bar{\mathcal{E}}}(\nu)}, \nu)$ and $g^{\bar{q}}_{\bar{\gamma}}(\nu) = \ell(\alpha_{\bar{o}^{\bar{\mathcal{E}}}(\nu)}, p_{\bar{o}^{\bar{\mathcal{E}}}(\nu)}, \nu)$.

Now suppose that $\alpha < \tau$ and $p \in \mathcal{R}_\lambda$. Take $p' \leq p \restriction \delta$ in Y_α and let ι be such that $\alpha_\iota = \alpha$ and $p_\iota = p'$. Then it is forced in $\mathcal{R}_\lambda$ that $\bar{q}$ forces in $Q^{\bar{\mathcal{E}}}_\xi$ that there is some $q' \in G_\xi$ with $\nu^{q'}_{\bar{\gamma}} = f^{q}_{\bar{\gamma}}(\nu) = k(\alpha, p', \nu)$ and $g^{q'}_{\bar{\gamma}}(\nu) \restriction \xi = \ell(\alpha, p', \nu)$ in H_ξ. It follows that $k(\alpha, p', \nu) \in C_{\xi,\bar{\gamma}} \cap D_\alpha$. $\square$

Now let $i\colon V \to \mathrm{ult}(V, \bar{U}(\beta)) = M$ [[Notation????]] and Let $\bar{Q}$ be $(Q^{\mathcal{E}}_\lambda)^M$. By the choice of g',

$$M \models \Vdash_{\mathcal{R}^{\mathcal{E}}_{\lambda+1}} i(g')(\lambda) \parallel_{\mathcal{R}^{i(\mathcal{E})}_{\lambda+1,i(\lambda)+1}} \sigma.$$

Let σ' be the sentence $i(g')(\lambda) \Vdash_{\mathcal{R}_{\lambda+1,i(\lambda)+1}} \sigma$. Since $\mathcal{E}$ is incomplete in M, so we can apply case 1 there to the sentence σ'.

q in $(Q^{\mathcal{E}}_\lambda)^{(M)}$ so that

$$M \models \Vdash_{\mathcal{R}_\lambda} r \parallel_{Q^{\mathcal{E}}_\lambda} i(g')(\lambda) \Vdash_{\mathcal{R}^{i(\mathcal{E})}_{\lambda+1,i(\lambda)+1}} \sigma,$$

and thus it is forced in $\mathcal{R}_\lambda$ that r, together with $i(g')(\lambda)$, decides σ.

By using Claim 3.38 we can regard σ, below add(q', γ, ν) for any $\nu \in A$, as a sentence in $\mathcal{R}^{\bar{\mathcal{E}}}_{\nu+1}$ where $\bar{\mathcal{E}} = \mathcal{E}{\downarrow}\nu$. We now need to determine, for $\nu \in A$, the condition $q'{\downarrow}\nu \in Q^{\bar{\mathcal{E}}}\nu$. It is tempting to simply use the fact, which we know

as an induction hypothesis, that $\mathcal{R}^{\bar{\mathcal{E}}}_{\nu+1}$ has the Prikry property, but such an argument breaks down because we have no control over what happens in $\mathcal{R}_{\xi+1}$, an incompacity which is determined by the use of a diagonal intersection in the proof of lemma 3.33.

Claim 3.40. *There is a $\leq^*$-dense set of conditions r in $Q^{\bar{\mathcal{E}}}_{\nu}$ such that the following property is forced in $\mathcal{R}_{\nu}$: For all pairs (γ', ν') with $\gamma' \propto_{\nu'} \eta$ and $\xi < \nu' < \nu$, either $r_{*\gamma',\nu'} \Vdash \sigma$ or else there is no $r' \leq^* r$ such that $r'_{*\gamma',\nu'} \Vdash \sigma$.*

Proof. Use Lemma 3.37 to define a $\leq^*$ descending sequence of conditions r^{ι} such that r^{ι} has the required property for pairs (γ', ν') with $\nu' < \iota$, and then use the diagonal intersection of this sequence.

The fact that the diagonal intersection inherits the required property for each pair (γ', ν') from r^{ι} follows from the observation that this property depends only on r/ν'. This is because $r_{*\gamma',\nu'}\restriction\gamma' + 1$ is the maximal condition, so that r restricts only extensions on the interval between γ' and γ. However any $r' \leq r$ with $\nu^{r'}_{\gamma''} > \nu^{r}_{\gamma''}$ must have $\nu^{r'}_{\gamma''} > \nu'$. $\square$

It might be noted that Claim 3.40 would still be true without the restriction $\xi < \nu'$; however this observation is not relevant since only r/ξ will survive the diagonal intersection used in the proof of Lemma 3.33.

Now choose, for each $\nu \in A$, a condition $r_{\nu} \leq^* \downarrow\nu$ satisfying the conclusion of Claim 3.40. By the usual methods, we can see that

Claim 3.41. *There is $q' \leq^* q$ and $A' \subseteq A$ with $A' \in \bar{U}_{\beta}$ [[notation????]] such that*

1. *For each $\nu \in A'$, $r_{\nu} = q'\downarrow\nu$.*
2. *If $\nu_0, \nu_1 \in A'$ then it is forced in $\mathcal{R}_{\nu}$ that $r_{\nu_0} \parallel_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash \mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}\sigma$ if and only if $r_{\nu_1} \parallel_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash \mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}\sigma$.*

$$r_{\nu_0} \Vdash_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash_{\mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}} \sigma \iff r_{\nu_1} \Vdash_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash_{\mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}} \sigma,$$

and

$$r_{\nu_0} \Vdash_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash_{\mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}} \neg\sigma \iff r_{\nu_1} \Vdash_{Q^{\mathcal{E}_{\nu}}_{\nu}} g'(\nu) \Vdash_{\mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}} \neg\sigma.$$

[[[Here g' is the function defined previously.]]]

This completes the definition of the condition $q' \leq^* q$ in case 1, and it only remains to verify that it is as required. We are given $q'' \leq^*$ such that $q''/\xi \leq^* q'$ and $A^{q''}_{\gamma} \cap \xi + 1 = \emptyset$, and a condition $q''' \leq q''$ which decides σ. Suppose that $q''' \Vdash \sigma$. By extending q''' if necessary, we may assume that $\operatorname{add}(q'', \gamma, \nu) \leq q'''$ for some $\nu \in A'$, and by the choice of $g'(\nu)$ we may then suppose that $q''' = \operatorname{add}(q'', \gamma, \nu)$ and that

Damn. I'm fogetting that $q'{\downarrow}\nu$ will have $\beta_{\bar\gamma} = \bar\beta = o^{\bar{\mathcal{E}}}(\nu)$. This means that the case $\gamma + 1 = \operatorname{len}(\mathcal{E})$ and $\mathcal{E}$ is not complete comes into action. It seems that $\mathcal{E}$ incomplete should be case 1, and the current case should be case 2 with the argument running like this: Use Claim 3.38 to reduce to case 1, with $o^{\mathcal{E}}(\lambda) = \beta + 1$.

3.4 The Prikry Property for $\operatorname{len}(\mathcal{E})$ a limit ordinal

I think I want to change this combine $\operatorname{len}(\mathcal{E})$ a limit ordinal with complete $\mathcal{E}$ of successor length. Some rough notes follow:

We say that an extension $q' \le q$ is *γ-bounded* if $\nu^{q'}_{\gamma'} = \nu^{q}_{\gamma'}$ for all $\gamma' \ge \gamma$. (???)It is *strongly γ-bounded* if $q_{\gamma'} = q'_{\gamma'}$ for all $\gamma' \ge \gamma$.

An extension $q' \le q$ is *above* ξ if for all $\gamma < \operatorname{len}(\mathcal{E})$, $A^{q'}_\gamma \cap \xi = A^{q}_\gamma \cap \xi$ and $g^{q'}_\gamma(\nu) = g^{q}_\gamma(\nu)$ for all $\nu \in A^q_\gamma \cap \xi$.

Note that if $\delta \le \lambda$ and $\vec q = \langle\, q^\xi : \xi < \delta \,\rangle$ is $\le^*$-decreasing, with $q^{\xi'} \le^* q^\xi$ being above ξ, then $\vec q$ has a lower bound $\bigwedge_{\xi<\delta} q^\xi$.

However this seems to require more, because of the requirement that $\beta^{q^{\xi'}_\gamma} \propto_\nu \beta^{q^{\xi}_\gamma}$ for all $\nu \in A^\xi_\gamma$.

11/13/07 — All this needs to be changed. The critical point is that since the extension needs to be above ξ, I can't really do anything about class 3 and 4 extensions *below* ξ. However I can define $g^{q'}_\gamma(\nu)$ so that it (working above ξ) decides, for each γ' and ν' with $\xi < \nu' < \nu$, whether $q'' := \operatorname{add}(q'_{*\gamma',\nu'}, \gamma, \nu)$ decides (with no input from $q''{\restriction}\gamma'+1$) the sentence σ. This, in fact, has already been done in Claim 3.50. This argument will be enough to show that there is a $\gamma' + 1$-bounded condition deciding σ.

I haven't quite figured out how to deal with γ. I'm thinking that I might end up needing a stronger version of the Prikry theorem for incomplete sequences $\mathcal{E}$.

The main lemma is this:

Lemma 3.42. *Suppose either $\gamma + 1 < \operatorname{len}(\mathcal{E})$ or $\gamma + 1 = \operatorname{len}(\mathcal{E})$ and $\mathcal{E}$ is complete, and fix $\xi < \lambda$. Then for any condition q there is $q' \le^* q$, with the extension being above ξ, such that for any $\gamma+1$-bounded $q'' \le q'$ deciding σ there is γ-bounded $q''' \le q'$ such that $q'' \le q''' \le q'$ and $q''' \parallel \sigma$.*

Corollary 3.43. *Suppose that $\mathcal{E}$ is complete, $q \in Q^{\mathcal{E}}_\lambda$, and σ is any sentence of the forcing language. Then there is $q'' \le^* q$ such that $q'' \parallel \sigma$.*

Proof. Using Lemma 3.42, define a $\le^*$-descending sequence $\langle\, q^\xi : \xi \le \lambda \,\rangle$ such that (i) $q^{\xi'} \le^* q^\xi$ is $\xi{+}1$ bounded whenever $\xi < \xi' < \lambda$, and (ii) $q^{\xi+1}$ satisfies the conclusion of Lemma 3.42 whenever $\gamma \propto_\xi \operatorname{len}(\mathcal{E})$ and γ satisfies the hypothesis. Clause (i) implies that we can set $q^\xi = \bigwedge_{\xi'<\xi} q^{\xi'}$ for limit ordinals $\xi \le \lambda$. The fact that $\{\, \gamma : \gamma \propto_\xi \operatorname{len}(\mathcal{E}) \,\}$ has size at most ξ implies that Lemma 3.42 can be used to define $q^{\xi+1}$ satisfying clause (ii).

We claim that $q' = q^\lambda$ is as required. To see this, let γ' be the least ordinal such that there is γ'-bounded $q'' \leq q'$ deciding σ. Then γ' cannot be a limit ordinal. If γ' is a successor ordinal, $\gamma' = \gamma + 1$, then $q' \leq q^\lambda \leq^* q^{\xi+1}$ where $\gamma \propto_\xi \operatorname{len}(\mathcal{E})$, and it follows from the choice of ξ that $q'' \leq q'$ is γ-bounded, contradicting the minimality of γ'.

It follows that $q'' \leq q'$ is 0-bounded, but this means that $q'' \leq^* q' \leq^* q$, so that q'' is as required. □

Sketch of proof of 3.42. We will set $\beta^{q'}_\gamma = \beta := \beta^q_\gamma + 1$. We will have $A^{q'}_{\gamma,\beta} \cap \xi + 1 = \emptyset$.

==

First we define $g^{q'}_\gamma$, as well as $q' \lfloor \gamma + 1$, and make a preliminary further extension of q_γ, to obtain a condition q'' with the property that if $\nu \in A^{q''}_{\gamma,\beta}$ then it is forced in $\mathcal{R}_{\nu+1}$ that either $\bar{q} := \operatorname{add}(q'', \gamma, \nu) \parallel \gamma$ or else there is no $\gamma + 1$-bounded $q''' \leq \bar{q}$ which decides σ. Here we are treating the forcing as $\mathcal{R}_{\nu+1} \times \mathcal{R}^{\mathcal{E}}_{\nu+1,\lambda+1}$ and in particular we do not consider whether $\operatorname{add}(q'', \gamma, \nu) \leq q''$.

This involves extending $q \lfloor \gamma + 1$ with the extension above ν, and also extending $g_\gamma(\nu)$ and $q_\gamma/\nu + 1$

This may involve the distributivity of $Q^{\mathcal{E}}_\lambda$. N.B. — I can probably get everything decided by a Definition 3.17(4,5) extension, so I wouldn't have to change q_γ other than $f_\gamma \upharpoonright A_{\gamma,\beta}$ and $g_\gamma \upharpoonright A_{\gamma,\beta}$.

In the second step we look at what we now have for $\bar{q} := \operatorname{add}(q', \gamma, \nu)$ and extend the condition in $Q^{\mathcal{E}\upharpoonright\gamma+1\downarrow\nu}_\nu$ which it implies so that for each condition ${}_{*\gamma',\nu'}$, either the condition $\operatorname{add}({}_{*\gamma',\nu'}, \gamma, \nu)$ decides σ or else there is no direct extension of $q' \lfloor \gamma' + 1$ which would make it do so.

After doing this for each ν, we shrink $A^{q'}_{\gamma,\beta}$ so that all νs in this set agree on all of this.

To see that this condition satisfies the conclusion of Lemma 3.42, we suppose that $q'' \leq q'$ is $\gamma + 1$-bounded and q'' decides σ. Since $P^*_{???}$ [[What was this called?]] is dense in $\mathcal{R}_\nu$, we can use the result of the first step to see that we can assume *wlog* that $q'' = \operatorname{add}(q'_{*\gamma',\nu'}, \gamma, \nu)$ for some relevant γ', ν'. But then we would have gotten the same result with $\operatorname{add}(q'_{*\gamma',\nu'}, \gamma, \bar{\nu})$ for any $\bar{\nu}$ in $A^{q'}_{\gamma,\beta}$. But $q'_{*\gamma',\nu'}/\xi + 1$ forces that there is some such condition in the generic set, so $q'_{*\gamma',\nu'}/\xi + 1$ already decides σ.

Note: where I have $q'_{*\gamma',\nu'}$, add "or q' itself." □

This is the end of the proposed new proof. The rest of this subsection, and all of the next, is old and (hopefully) is obsolete.

The proof of lemma 3.42 is in two (maybe three) steps.

In this and the following two subsections we use the induction hypothesis that $\mathcal{R}_\lambda$ satisfies the Prikry property to show that $\mathcal{R}_{\lambda+1}$ does so. For this it will be sufficient to show that $Q^{\mathcal{E}}_\lambda$ satisfies the Prikry property:

Proposition 3.44. *Suppose that $\mathcal{R}_\lambda$ satisfies the Prikry property and for all sentences σ and conditions $q \in Q^{\mathcal{E}}_\lambda$, there is $q' \leq^* q$ such that $\Vdash_{\mathcal{R}_\lambda} q' \parallel_{Q^{\mathcal{E}}_\lambda} \sigma$. Then $\mathcal{R}_{\lambda+1}$ has the Prikry property.*

Proof. Suppose $p \in \mathcal{R}_{\lambda+1}$. We may assume that $\lambda \in \operatorname{domain}(p)$, since if it is not then any extension $p' \leq p$ with $p' \restriction \lambda = p \restriction \lambda$ satisfies $p' \leq^* p$. Suppose $p_\lambda = (\mathcal{E}, q)$ and let $q' \leq q$ in $Q^{\mathcal{E}}_{\lambda}$ be as in the hypothesis. Since $\mathcal{R}_\lambda$ satisfies the Prikry property, there is $p' \leq^* p \restriction \lambda$ so that $p' \parallel_{\mathcal{R}_\lambda} q' \Vdash_{Q^{\mathcal{E}}_{\lambda}} \sigma$. Then $p' {}^\frown \langle \lambda, (\mathcal{E}, q') \rangle \parallel_{\mathcal{R}_{\lambda+1}} \sigma$. □

The proof that $Q^{\mathcal{E}}_{\lambda}$ satisfies the Prikry property will be by induction on $\operatorname{len}(\mathcal{E})$: In addition to the assumption that $\mathcal{R}_\lambda$ satisfies the Prikry property, we will assume throughout that for each $\gamma < \operatorname{len}(\mathcal{E})$ the forcing $Q^{\mathcal{E}\restriction\gamma}_{\lambda}$ satisfies the Prikry property.

The end result of this subsection is the limit case:

Lemma 3.45. *Suppose that $\eta = \operatorname{len}(\mathcal{E})$ is a limit ordinal. Then for each $q \in Q^{\mathcal{E}}_{\lambda}$ there is $q' \leq^* q$ such that $\Vdash_{\mathcal{R}_\lambda} q' \parallel_{Q^{\mathcal{E}}_{\lambda}} \sigma$.*

We actually prove a slightly more general result which will be used in the following two subsections to show if $\operatorname{len}(\mathcal{E})$ is a successor then we only need to consider the last coordinate of a condition q:

Lemma 3.46. *For any $q \in Q^{\mathcal{E}}_{\lambda}$ there is a condition $q' \leq^* q$ such that the following is forced in $\mathcal{R}_\lambda$:*

> *Either $q' \parallel_{Q^{\mathcal{E}}_{\lambda}}$, or else $\operatorname{len}(\mathcal{E})$ is a successor ordinal $\gamma + 1$ and any condition $q'' \leq q'$ such that $q'' \parallel_{Q^{\mathcal{E}}_{\lambda}} \sigma$ has $\nu^{q''}_{\gamma} > \nu^{q}_{\gamma}$.*

Notation 3.47. We write $q_{*\gamma,\nu}$ for $(t^{\mathcal{E}\restriction\gamma+1}_{\nu} \restriction \gamma + 1) {}^\frown (q \downharpoonleft \gamma + 1)$.

Note that $q_{*\gamma,\nu} = \operatorname{add}(q, \gamma, \nu)$ in the case that $\bar{o}^{\mathcal{E}\restriction\gamma+1}(\nu) > \beta^{q}_{\gamma}$. We will have $q_{*(}\gamma, \nu) \leq q$ whenever there is ν' such that $\nu^{q}_{\gamma} \leq \nu' \leq \nu$, and

$$q_{*\gamma,\nu} \leq \operatorname{add}^{\mathcal{E}}(q, \gamma, \nu') \leq q \tag{4}$$

with the first inequality in (4) holding by clause (4) of Definition 3.17 and the second inequality holding by clause (5). In particular $q_{*\gamma,\nu} \leq q$ typically does not holds unless $\mathcal{E}\restriction\gamma + 1$ is complete, which of course is the case for γ as in lemma 3.46.

The following observation is important because, while there are λ^+ many conditions in $Q^{\mathcal{E}}_{\lambda}$, there are only λ many conditions of the form $q_{*\gamma,\nu}$.

Proposition 3.48. *Suppose that $q' \leq q$ in $Q^{\mathcal{E}}_{\lambda}$, and $\operatorname{len}(\mathcal{E}) > \gamma + 1$, where $\gamma = \max\left\{ \gamma' : \nu^{q'}_{\gamma'} > \nu^{q}_{\gamma'} \right\}$. Then q' forces that $q_{*\gamma,\nu} \in G_\lambda$ for some $\nu \geq \nu^{q'}_{\gamma}$.*

Proof. Suppose $q'' \leq q'$. If $\nu^{q''}_{\gamma'} = \nu^{q}_{\gamma'}$ for all $\gamma' > \gamma$ then there is ν such that $q''_{*\gamma,\nu} \leq q''$, and then $q''_{*\gamma,\nu} \leq q_{*\gamma,\nu}$. Hence we can suppose that there are $\gamma'' > \gamma$ such that $\nu^{q''}_{\gamma''} > \nu^{q}_{\gamma''}$, and this implies that there are $\gamma' > \gamma$ and $\nu' > \nu^{q'}_{0}$ such that $q'' \leq \operatorname{add}(q', \gamma', \nu') \leq q'$. Then $\gamma \propto_{\nu'} \gamma'$ since $\{\vec{\nu}', \gamma'\}\downarrow\nu'$ exists. If we set $\bar{\gamma} = \gamma\downarrow\nu'$ then $\mathcal{E}_{\nu'} \restriction \bar{\gamma} + 1 = (\mathcal{E}_\lambda \restriction \gamma + 1)\downarrow\nu'$ is complete, so there is some ν such that $(q'\downarrow\nu')_{*(}\bar{\gamma}, \nu) \leq (q'' | \gamma + 1)\downarrow\nu'$ in $Q^{\mathcal{E}_{\nu'}\restriction\bar{\gamma}+1}_{\nu'}$. Then $q_{*\gamma,\nu}$ is compatible with q''. □

Proposition 3.49. *Suppose that $\dot D$ is a $\mathcal{R}_\lambda$-name for a dense subset of $(Q^{\mathcal{E}}_\lambda, \leq^*)$, and let $D' = \left\{ q \in Q^{\mathcal{E}}_\lambda : \Vdash_{\mathcal{R}_\lambda} q \in \dot D \right\}$. Then D' is dense in $(Q^{\mathcal{E}}_\lambda, \leq^*)$.*

Proof. Fix $q \in Q^{\mathcal{E}}_\lambda$, and define a $\leq^*$-decreasing sequence of conditions $\langle q^\nu : \nu < \delta \rangle$, with $q^0 \leq^* q$, in $Q^{\mathcal{E}}_\lambda$, and an antichain of conditions $\langle p^\nu : \nu < \delta \rangle$ in $\mathcal{R}_\lambda$ such that $p^\nu \Vdash_{\mathcal{R}_\lambda} q^\nu \in \dot D$ for each $\nu < \delta$. This construction must stop at some stage $\delta < \lambda$ since $\mathcal{R}_\lambda$ satisfies the λ-chain condition. On the other hand this construction can always be continued until $\{ p^\nu : \nu < \delta \}$ is a maximal antichain: otherwise let $\bar q = \bigwedge_{\nu<\delta} q^\nu$, and suppose that p is incompatible with p^ν for all $\nu < \delta$. By the hypothesis there is $p' \leq p$ and $\bar q' < q$ such that $p' \Vdash q' \in \dot D$, so we could have taken $q^\delta = \bar q'$ and $p^\delta = p'$.

Finally set $q' = \bigwedge_{\nu<\delta} q^\nu$. Then $\Vdash_{\mathcal{R}_\lambda} q' \in \dot D$. $\square$

Claim 3.50. *Let $\mathcal{X}^\sigma_0$ be the set of $q \in Q^{\mathcal{E}}_\lambda$ such that for each $\gamma + 1 < \eta$ the following sentence is forced in $\mathcal{R}_\lambda$: If q' is any condition in $Q^{\mathcal{E}}_\lambda$ such that*

$$q' \leq q, \qquad q' \Vdash \sigma, \quad \text{and} \quad q' \lfloor \gamma + 1 \leq^* q \lfloor \gamma + 1 \tag{5}$$

then $(q' \upharpoonright \gamma + 1)^\frown (q \lfloor \gamma + 1) \Vdash \sigma$.

Then $\mathcal{X}^\sigma_0$ is dense in $(Q^{\mathcal{E}}_\lambda, \leq^*)$.

Proof. Let $q^0 \in Q^{\mathcal{E}}_\lambda$ be arbitrary; we will find a condition $q \leq^* q^0$ in $\mathcal{X}^\sigma_0$. If there is any condition $q \leq^* q^0$ such that $q \Vdash \sigma$ then we can $q \in \mathcal{X}^\sigma_0$. For the rest of the proof we assume that no $q \leq^* q^0$ forces σ.

Define a $\leq^*$-decreasing sequence $\langle q^\nu : \nu < \lambda \rangle$ below q^0 such that each condition $q^{\nu+1}$ satisfies the following property:

$$\forall q' \leq^* q^\nu \, \forall \gamma \propto_\nu \eta \, \left(q'_{*\gamma,\nu} \Vdash \sigma \implies q^{\nu+1}_{*\gamma,\nu} \Vdash \sigma \right). \tag{6}$$

9/3/07 *** $(Q^{\mathcal{E}}_\lambda, \leq^*)$ is, in fact, $< \lambda$-closed, the only need for comment being that to find $\bigwedge_{\nu<\xi} q^\nu$ we will have to replace q^ν with q^ν/ξ. This is ok, since $q^\nu/\xi \leq^* q^\nu$ and since $\leq^*$ is preserved by $q \mapsto q/\xi$. This is necessary because the conditions $g^q(\nu')$ are only $< \nu'$-closed.

Has this been stated somewhere?

Since $(Q^{\mathcal{E}}_\lambda, \leq^*)$ is $<\lambda$-closed, it will be sufficient to show how to choose $q^{\nu+1}$, given q^ν. Since there are only ν many ordinals $\gamma \propto_\nu \eta$, it will be sufficient to show that the set of conditions which satisfy (3) for fixed γ and ν is $\leq^*$-dense in $Q^{\mathcal{E}}_\lambda$. Now certainly it is forced in $\mathcal{R}_\lambda$ that this set is dense, since if $q' \leq^* q$ and p is any condition in $\mathcal{R}_\lambda$ such that $p \Vdash_{\mathcal{R}} q'_{*\gamma,\nu} \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$, then p also forces that $q' \leq^* q$ satisfies (3) for this γ and ν. It follows by proposition 3.49 that the set of conditions q such that it is forced in $\mathcal{R}_\lambda$ that q satisfies (3) for this γ and ν is $\leq^*$-dense.

Now set $q = \Delta_\nu q^\nu$. Then if $q' \leq^* q$ and $q'_{*\gamma,\nu} \Vdash \sigma$ then $q_{*\gamma,\nu} \Vdash \sigma$ since $q'_{*\gamma,\nu} \leq^* q_{*\gamma,\nu} \leq^* (q^\nu/\nu)_{*\gamma,\nu} = q^\nu_{*\gamma,\nu}$.

Now suppose that $q' \leq q$ and $q' \Vdash \sigma$. Then $q' \not\leq^* q$ since by assumption $q' \not\leq^* q^0$, so let $\gamma = \max \left\{ \gamma' : \nu^{q'}_{\gamma'} > \nu^q_{\gamma'} \right\}$. By proposition 3.48, q' forces that

there is some $\nu \geq \nu^{q'}_\gamma$ such that $q'_{*\gamma,\nu} \in G_\lambda$. Then $q'_{*\gamma,\nu} \Vdash \sigma$ since $q'_{*\gamma,\nu} \leq q'$, and by the choice of q it follows that $q_{*\gamma,\nu} \Vdash \sigma$. $\square$

Definition 3.51. If $\gamma + 1 \leq \eta$ and σ is any sentence, then we write $q \parallel^{\gamma+1} \sigma$ if $q \in \mathcal{X}^\sigma_0$ and it is forced in $\mathcal{R}_\lambda$ that either $q \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$ or else

$$q|(\gamma+1) \Vdash_{Q^{\mathcal{E}\restriction\gamma+1}_\lambda} \neg\exists q' \in \dot{G}\ \left(q'^\frown(q\lfloor(\gamma+1)) \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma\right). \tag{7}$$

Note that $\dot{G}$ in (7) is a name for the generic subset of $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$.

Proposition 3.52. *For any $\gamma + 1 < \operatorname{len}(\mathcal{E})$ and any sentence σ there is a $\leq^*$-dense set of conditions q such that $q \parallel^{\gamma+1} \sigma$.*

Proof. By proposition 3.49 it will be sufficient to show that for all $p \in \mathcal{R}_\lambda$ and $q \in Q^{\mathcal{E}}_\lambda$ there are $p' \leq p$ and $q' \leq^* q$ such that p' forces that one of the alternatives in definition 3.51 holds.

Consider the following sentence in the forcing language for $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$:

$$\exists q' \in \dot{G}\ \big(q' \leq q|\gamma+1 \text{ and } (q'\restriction\gamma+1)^\frown(\bar{q}\lfloor\gamma+1) \Vdash \sigma\big). \tag{8}$$

Since $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$ satisfies the Prikry property by the induction hypothisis, there is $\bar{q}' \leq^* q|\gamma+1$ in $Q^{\mathcal{E}\restriction\gamma+1}_\lambda$ which decides the sentence (8). Now set $q' = (\bar{q}'\restriction\gamma+1)^\frown(q\lfloor\gamma+1)$.

If q' decides (8) negatively, then q' satisfies (7). Otherwise let $q'' \leq^* q'$ be such that for all for all $\gamma' > \gamma$, $\beta^{q''}_{\gamma'} > 0$, and if $\nu \in A^{q''}_{\gamma'}$ then $\gamma \propto_\nu \gamma'$ and the set of $\nu' < \nu$ such that $q_{*\gamma,\nu'} \Vdash \sigma$ is dense in $P^{(\mathcal{E}\restriction\gamma+1)\downarrow\nu}_\nu = P^{\mathcal{E}\restriction\gamma+1}_\lambda \cap \nu$. Then by proposition 3.48 (or at least by its proof), q'' forces that there is $\bar{q}''\downarrow\nu_{*\gamma,\nu} \in G_\nu$ such that $\bar{q}''_{*\gamma,\nu} \Vdash \sigma$. Hence $q'' \Vdash \sigma$. $\square$

In the case $\operatorname{cf}(\operatorname{len}(\mathcal{E})) < \lambda$ we could finish the proof at this point: let $\langle \gamma_\iota : \iota < \operatorname{cf}(\operatorname{len}(\mathcal{E})) \rangle$ be cofinal in $\operatorname{len}(\mathcal{E})$ and let $q = \bigwedge_{\iota<\operatorname{cf}(\operatorname{len}(\mathcal{E}))} q^\iota$ where $\langle q^\iota : \iota < \operatorname{cf}(\operatorname{len}(\mathcal{E})) \rangle$ is a $\leq^*$-descending sequence of conditions such that $q^\iota \parallel^{\gamma_\iota+1} \sigma$ and $q^\iota \parallel^{\gamma_\iota+1} \neg\sigma$.

Now suppose that $q' \leq q$ is a condition such that $q' \parallel \sigma$; say $q' \Vdash \sigma$. Pick $\iota < \operatorname{cf}(\operatorname{len}(\mathcal{E}))$ so that $\gamma_\iota > \max\left\{\gamma : \nu^{q'}_\gamma > \nu^q_\gamma\right\}$. Then $q' \leq q^\iota$, and it follows that $q^\iota \Vdash \sigma$. Since $q \leq^* q^\iota$ it follows that $q \Vdash \sigma$.

In the case $\operatorname{cf}(\operatorname{len}(\mathcal{E})) = \lambda$, however, The limit $\bigwedge_{\iota<\lambda} q^\iota$ does not, in general, exist. The next definition gives a criterion on the sequence which implies that $\bigwedge_{\iota<\lambda} q^\iota$ does exist:

Definition 3.53. Using induction on $\operatorname{len}(\mathcal{E})$, we say that conditions $q, q' \in Q^{\mathcal{E}}_\lambda$ *agree up to* $\nu < \lambda$ if, for all $\gamma < \operatorname{len}(\mathcal{E})$, $A^q_\gamma \cap \nu = A^{q'}_\gamma \cap \nu$ and for all $\nu' \in A^q_\gamma \cap \nu$, $g^q_\gamma(\nu')$ agrees with $g^{q'}_\gamma(\nu')$ up to ν in $Q^{\mathcal{E}\restriction\gamma}_\lambda$.

We will say that a sequence $\langle q_\nu : \nu < \xi \rangle$ of length $\xi \leq \lambda$ is steadily $\leq^*$-decreasing if for each $\nu < \nu' < \xi$, $q_{\nu'} \leq^* q_\nu$ and $q_{\nu'}$ agrees with q_ν up to $\nu + 1$.

Proposition 3.54. *Suppose that $\langle q_\nu : \nu < \xi \rangle$ is a steadily $\leq^*$-descending sequence of conditions in Q^ξ_λ of length $\xi \leq \lambda$ then the sequence has a lower bound $q = \bigwedge_{\nu<\xi} q_\nu$ such that for each $\nu < \xi$, q agrees with q_ν up to $\nu + 1$.* □

Note that the lower bounds given by Proposition 3.54 are essentially a disguise for diagonal limits, so that their use in the following lemma is more delicate than the case $\mathrm{cf}(\eta) < \lambda$. Note that in the case that η is a limit ordinal, lemma 3.55 is the same as Lemma 3.46.

Lemma 3.55. *The set $\mathcal{X}^\sigma_1$ is dense in $(Q^\xi_\lambda, \leq^*)$, where $\mathcal{X}^\sigma_1$ is the set of conditions $q \in Q^\xi_\lambda$ such that it is forced in $\mathcal{R}_\lambda$ that either $q \parallel \sigma$, or else η is a successor ordinal and there is no $q' \leq q$ such that $\nu^{q'}_{\eta-1} = \nu^q_{\eta-1}$ and $q' \parallel \sigma$.*

Proof. Fix $q^0 \in Q^\xi_\lambda$. We may assume that $q^0 \in \chi^\sigma_0 \cap \chi^{\neg\sigma}_0$, and that if $\gamma < \eta$ then $\beta^{q^0}_\gamma > 0$ and $\gamma \propto_\nu \eta$ for all $\nu \in A^{q^0}_\nu$.

We will define by recursion on ν a stably $\leq^*$-decreasing sequence $\langle q^\nu : \nu \leq \lambda \rangle$ of conditions in Q^ξ_λ. At limit ordinals ν we will take $q^\nu = \bigwedge_{\nu'<\nu} q^{\nu'}$, so that in particular $q^\lambda \leq^* q^\nu$ for each $\nu < \lambda$. We are given q^0, and by Proposition 3.55 we can take $q^\nu = \bigwedge_{\nu'<\nu} q^{\nu'}$ for limit ordinals ν. In the rest of the proof we will define $q^{\nu+1}$, assuming that q^ν has alread been defined.

Say that an extension $q' \leq q$ is γ-bounded if $\nu^{q'}_{\gamma'} = \nu^q_{\gamma'}$ for all $\gamma' > \gamma$, and say that the extension $q' \leq q$ is γ, ν-bounded if in addition $\vec{\nu}^{q'}{\downarrow}\nu$ exists.

For each successor ordinal $\nu + 1 < \lambda$ we will have

1. Suppose that $\gamma \propto_\nu \eta$ and $q \leq q^{\nu+1}$ is γ, ν-bounded. Then either $q \parallel \sigma$ or else there is no γ, ν-bounded condition $q' \leq q/\nu + 1$ such that $q' \parallel \sigma$.

Since there are only $|\nu|$-many ordinals $\gamma \propto_\nu \eta$, it will be sufficient to show that for any fixed $\gamma \propto_\nu \eta$ and any condition q_0 we can find $q_1 \leq^* q_0$ such that q_1 and q_0 agree up to ν and q_1 satsifies the condition required of $q^{\nu+1}$ for the single ordinal γ.

Since $q^0 \in \chi^\sigma_0 \cap \chi^{\neg\sigma}_0$, we can take

2. $q^\nu{\restriction}\gamma + 1 = q^0{\restriction}\gamma + 1$.

I need to say something about $\mathcal{R}_\lambda$. This is actually supposed to be proving that $\Vdash_{\mathcal{R}_\lambda}$ clause 3. This means I can ignore things like direct extensions below ν in the course of making the extensions $q \leq q^\nu$, but this needs to be explained.

We are given q^0 and for limit ν we have $q^\nu = \bigwedge_{\nu'<\nu} q^{\nu'}$, so we only need to show how to define $q^{\nu+1}$, given q^ν.

Now the set of conditions $q \leq^* q^\nu$ above ν is ν-closed, and since there are at most ν-many ordinals $\gamma \propto_\gamma \eta$, it will be enough to show how to satisfy clause 1 for a single ordinal γ.

Now consider the condition q of clause 1. Since $q \leq q^{\nu+1}$ will imply that $q \leq q^\nu$, we can look at the set of ν, γ-bounded $q \leq q^\nu$. There are more than ν many such, as the ν, γ-boundedness does not restrict $\leq^*$-extensions; however there are only ν-many sequences $\vec{\nu}^q$ among such extensions.

Suppose that $q' \leq q$ is a γ, ν-bounded extension. Then by Definition 3.17 there is a sequence

$$q' = q_n \leq q_{n-1} \leq \cdots \leq q_1 \leq q_0 = q \tag{9}$$

such that if $i < n$ then $q_{i+1} \leq q_i$ by one of clauses 1 or 3–5 of Definition 3.17. That is, either (i) $q_{i+1} \leq^* q_i$ (Clause 1), (ii) $q_{i+1} = \operatorname{add}^{\mathcal{E}}(q, \gamma', \nu') \leq q_i$ (Clause 3), or (iii) $q_{i+1} = q_{*\gamma',\nu'}$ (Clause 4 or 5). In the 2nd and 3rd case, we have $\gamma' < \gamma$, $\gamma' \propto_\nu \gamma$, and $\nu' < \nu$.

Our aim is to find, for any condition $r \in Q^{\mathcal{E}}_\lambda$, a condition $r' \leq^* r$ so that if if $q \leq r$ is a γ, ν-bounded extension by a sequence as above, and if q is compatible with r', then (i) the 3rd case (Clauses 4,5 of Definition 3.17) does not occur in the sequence, and (ii) it is forced in $\mathcal{R}_\lambda$ that either $q \wedge r' \parallel \sigma$ or else there is no γ-bounded $q' \leq (q \wedge r')/\nu + 1$ such that $q' \parallel \sigma$.

The proof is an induction on the length of the sequence by which $q \leq r$ was obtained. The induction step depends on the following two observations:

Proposition 3.56. *1. For any condition q, there is an extension $q' \leq^* q$ above ν such that $q_{*\gamma',\nu'}$ is incompatible with q' for all $\nu' < \nu$ and $\gamma' < \gamma$.*

2. For any γ, ν-bounded extension $q' = \operatorname{add}(q, \gamma', \nu')$ and any extension $r' \leq^* q'$ above ν, there is $r \leq^* q$ above ν such that if $r' = \operatorname{add}(r, \gamma', \nu') \leq r$ then $r' \leq^* q'$.

Proof. For Clause 1, define r' by choosing $\beta^{r'}_{\gamma'} > \beta^r_\gamma$ with $\lambda > \operatorname{cf}(\beta^{r'}_{\gamma'}) > \nu$. This will ensure that there is no extension $r'_{*\gamma',\nu'} \leq r'$ with $\nu' < \nu$ since any extension using Definition 3.17(4) requires that $\beta^{r'}_{\gamma'}{\downarrow}\nu'$ exists, and any extension using Definition 3.17(5) would require a previous extension by Definition 3.17(4.

For Clause 2, define $r' \lfloor \gamma' = q' \lfloor \gamma'$, and $(g^{r'}_{\gamma'}(\nu'))_\lambda/\nu + 1 = q' \upharpoonright \gamma'/\nu + 1$. $\square$

Now consider a γ, ν-bounded extension $q' \leq q$ by a sequence like (9) such that only the 2nd type of extensions occur, that is, $q_{i+1} = \operatorname{add}(q_i, \gamma'_i, \nu'_i)$ for each $i < n$. By using Clause 1 of Proposition 3.56 we can find $r_n \leq^* q_n$ so that there is no γ, ν-bounded extension of r_n of the 3rd type. By using the Prikry property for $Q^{\mathcal{E}\upharpoonright\gamma}_\lambda$, together with the assumption that $q^0 \in \chi^\sigma_0 \cap \chi^{\neg\sigma}_0$, we can also make r_n satisfy that either $r_n \parallel \sigma$ or else there is no γ-bounded extension of $r_n/\nu + 1$ which decides σ. By repeated use of Clause 2 of Proposition 3.56 we can find $r_i \leq^* q_i$ so that either r_i is incompatible with q_n or else $r_i \wedge q_n \leq^* r_n$. [[Do I really have the possibility that $r_i \wedge q_n = \mathbf{0}$?]]

[[Maybe better to throw in all $\gamma \propto_\nu \eta$ at this point rather than fix one γ earlier?]]

In particular, $r_0 \leq q_0$ has this property. Since there are only ν-many γ, ν-bounded extensions of q using only the 2nd type of extension, we can find an extension $r \leq^* q$ above ν which has the property for all such extensions. I claim that this extension has the desired property for all γ, ν-bounded extensions.

First, it is certainly true that no γ, ν-bounded $r' \leq r$ can include any type 3 extensions in its sequence, so we only need to consider extensions whose sequences involve only steps of type 1 and 2. [[[...]]] $\square$

=========== DELETE THE REST OF THE SUBSECTION

This next result is the main result of the section. I need to change references to point to this.

Lemma 3.57. *There is a $\leq^*$-dense set of conditions q with the property that $p \Vdash_{\mathcal{R}_\lambda} q \parallel_{Q^{\mathcal{E}}_\lambda} \sigma$ for any condition $p \in \mathcal{R}_\lambda$ with the property that there exists $q' \in Q^{\mathcal{E}}_\lambda$ such that $p \Vdash (q' \leq q \wedge q' \parallel \sigma)$ and, if $\operatorname{len}(\mathcal{E})$ is a successor, $\nu^{q'}_{\operatorname{len}(\mathcal{E})-1} = \nu^{q}_{\operatorname{len}(\mathcal{E})-1}$.*

Proof. Fix $q^0 \in Q^{\mathcal{E}}_\lambda$. Define a $\leq^*$-descending sequence q^ξ and an antichain of conditions p^ξ in $\mathcal{R}_\lambda$ as follows: suppose that the sequences have been defined for $\xi' < \xi$, let $q' = \bigwedge_{\xi'<\xi} q^{\xi'}$ and let p' be incompatible with $p^{\xi'}$ for all $\xi' < \xi$. If there is q'' and $p'' \leq p'$ so that $p'' \Vdash (q'' \leq q' \wedge q'' \Vdash \sigma)$ and $\nu^{q''}_{\operatorname{len}(\mathcal{E})-1} = \nu^{q}_{\operatorname{len}(\mathcal{E})-1}$ if $\operatorname{len}(\mathcal{E})$ is a successor, then there must be such q'' so that $q'' \leq^* q'$, as otherwise we could take q'' so that the least ι such that $\nu^{q''}_\iota > \nu^q_\iota$ is as small as possible, and get a contradiction from the previous lemma. Thus we can set $q^{\xi+1} = q''$ and $p^{\xi+1} = p''$.

Since $\mathcal{R}_\lambda$ has the λ-chain condition there must be some $\eta < \lambda$ at which $\{ p^\xi : \xi < \eta \}$ is a maximal antichain. Set $q = \bigwedge_{\xi<\eta} q^\xi$.

Now if p and q' are as in the statement, then the set of conditions $p \wedge p^\xi$ such that p is compatible with p^ξ is dense below p, and for any such ξ we have $p \wedge p^\xi \Vdash q^\xi \Vdash \sigma$. Hence q is as required. $\square$

Lemma 3.58. *If $\mathcal{E}$ is complete and $\eta = \operatorname{len}(\mathcal{E})$ is a limit ordinal then $\mathcal{R}^{\mathcal{E}}_{\lambda+1}$ satisfies the Prikry property.*

Proof. Let $q \in Q^{\mathcal{E}}_\lambda$ be any of the $\leq^*$-dense set of conditions which satisfy the conclusion of lemma ? for both of the sentences σ and $\neg\sigma$. Then $\Vdash_{\mathcal{R}_\lambda} q \parallel_{Q^{\mathcal{E}}_\lambda} \sigma$, as for any $p \in \mathcal{R}_\lambda$ there is $p' \leq p$ and $q' \leq q$ so that either $p' \Vdash_{\mathcal{R}_\lambda} q' \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$ or $p' \Vdash_{\mathcal{R}_\lambda} q' \Vdash_{Q^{\mathcal{E}}_\lambda} \neg\sigma$. Since $\operatorname{len}(\mathcal{E})$ is not a successsor, it follows from the choice of q that $p' \Vdash_{\mathcal{R}_\lambda} q \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$ or $p' \Vdash_{\mathcal{R}_\lambda} q \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$.

Now let p be any member of the $\leq^*$-dense set of conditions in $\mathcal{R}_\lambda$ such that $p \parallel_{\mathcal{R}\lambda} q \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$. Then $p^\frown\langle\lambda, (\mathcal{E}, q)\rangle \parallel_{\mathcal{R}^{\mathcal{E}}_{\lambda+1}} \sigma$. $\square$

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3.5 Suitability for non-complete $\mathcal{E}$ of successor length

Throughout this and the next section we assume that $\mathcal{E}$ is a sequence of successor length $\eta = \gamma + 1$. If the forcing $Q^{\mathcal{E}}_\lambda$ at λ were a true iterated forcing then $Q^{\mathcal{E}}_\lambda$ could be factored as $Q^{\mathcal{E}}_\lambda \equiv Q^{\mathcal{E}\restriction\gamma}_\lambda * \dot{Q}_\lambda$ where $\dot{Q}_\lambda$ is the forcing from [7] to add a single closed unbounded subset of λ. In this case we could quote the results of [7] directly. Unfortunately this factorization is only possible in the case that γ is not a limit ordinal, but the following proposition will nevertheless allow the use of the proofs from [7].

Proposition 3.59. *Suppose $q' \le q$ are conditions in $Q^{\mathcal{E}}_{\lambda}$ such that $q \in \mathcal{X}^{\sigma}_1$, $q' \Vdash \sigma$ and $\nu^{q'}_{\gamma} = \nu^{q}_{\gamma}$. Then there is $q'' \le^* q$ such that $q'' \Vdash \sigma$.* $\square$

Proof. If $\mu = \max\left\{\, \gamma' : \nu^{q'}_{\gamma'} > \nu^{q}_{\gamma'} \,\right\}$ then the hypothesis states that $\mu + 1 < \eta$. Thus we can use lemma 3.46 exactly as in the proof of lemma 3.45. $\square$

For the remainder of this subsection we additionally assume that $\mathcal{E}$ is not complete.

Claim 3.60. *Let $\mathcal{X}^{\sigma}_2$ be the set of $q \in \mathcal{X}^{\sigma}_1$ such that for all $\nu \in A^q_{\gamma}$*

1. $\mathrm{add}(q, \gamma, \nu) \in \mathcal{X}^{\sigma}_1$, *and*
2. *It is forced in $\mathcal{R}_{\nu+1}$ that one of the following conditions holds:*

$$g^q_{\gamma}(\nu){\restriction}\lambda \Vdash_{\mathcal{R}_{\nu+1,\lambda}} \qquad \mathrm{add}(q,\gamma,\nu) \Vdash_{Q^{\mathcal{E}}_{\lambda}} \sigma$$
$$g^q_{\gamma}(\nu){\restriction}\lambda \Vdash_{\mathcal{R}_{\nu+1,\lambda}} \forall q' \le \mathrm{add}(q,\gamma,\nu)\ \left(\nu^{q'}_{\gamma} = \nu \implies q' \nVdash_{Q^{\mathcal{E}}_{\lambda}} \sigma\right)$$

Then $\mathcal{X}^{\sigma}_2$ is dense in $(Q^{\mathcal{E}}_{\lambda}, \le^*)$.

Proof. By proposition 3.59, it will be sufficient to prove the apparently weaker version of claim 3.60 obtained by replacing "$\forall q' \le \mathrm{add}(q,\gamma,\nu)\ \left(\nu^{q'}_{\gamma} = \nu \implies q' \nVdash_{Q^{\mathcal{E}}{}_{\lambda}} \sigma\right)$" in clause 2 with $\forall q' \le^* \mathrm{add}(q,\gamma,\nu)\ q' \nVdash_{Q^{\mathcal{E}}_{\lambda}} \sigma$."

We will show that for any $\bar q \in \mathcal{X}^{\sigma}_1$ there is $q \le^* \bar q$ such that $q \in \mathcal{X}^{\sigma}_2$. We can assume without loss of generality that $\beta^{\bar q}_{\gamma} = \bar o^{\mathcal{E}}(\lambda)$. The condition q which we construct will differ from $\bar q$ only in its γth coordinate q_{γ}. Further, since $q \le^* \bar q$ we must have $\nu^q_{\gamma} = \nu^{\bar q}_{\gamma}$, $\beta^q_{\gamma} = \nu^{\bar q}_{\gamma} = \bar o^{\mathcal{E}}(\lambda)$, and $f^q_{\gamma} = f^{\bar q}_{\gamma}{\restriction}A^q_{\gamma}$; so that only the simple $\mathcal{E}{\restriction}\gamma$-term $(A^q_{\gamma}, g^q_{\gamma})$ remains to be specified. This term is defined by a recursion on ordinals $\nu < \lambda$: stage ν of the recursion defines a $\mathcal{E}{\restriction}\gamma$-term (A^{ν}, g^{ν}). At stage $\nu+1$ we will additionally determine whether $\nu \in A^q_{\gamma} = \Delta_{\nu<\lambda} A^{\nu}$ and, if so, specify the value of $g^q_{\gamma}(\nu)$. Finally, the definition of q is completed by setting $A^q_{\gamma} = \Delta_{\nu<\lambda} A^{\nu}$.

Set $(A^0, g^0) = (A^{\bar q}_{\gamma}, g^{\bar q}_{\gamma})$, and if ν is a limit ordinal then set $A^{\nu} = \bigcap_{\nu'<\nu} A^{\nu'} \setminus \nu$ and $g^{\nu}(\xi) = \bigwedge_{\nu'<\nu} g^{\nu'}(\xi)$ for each $\xi \in A^{\nu}$. This is possible since $\mathcal{R}^{\mathcal{E}}_{\bar\nu+1,\lambda+1}$ is $\bar\nu^{+}$-closed for each $\bar\nu \ge \nu$.

Now suppose that (A^{ν}, g^{ν}) and $g^q_{\gamma}{\restriction}\nu$ have been defined. If $\nu \notin A^{\nu}$ then set $A^{\nu+1} = A^{\nu}$ and $g^{\nu+1} = g^{\nu}$. Otherwise pick $p' \le^* g^{\nu}(\nu){\restriction}\lambda$ in $\mathcal{R}_{\nu+1,\lambda}$ and $q' \le^* (g^{\nu}(\nu))_{\lambda} {}^\frown \langle \gamma, (\nu, \beta^{\bar q}_{\gamma}, A^{\nu}, g^{\nu}, f^{\bar q}) \rangle$ in $(Q^{\mathcal{E}}_{\lambda}{\restriction}\eta+1) \cap \mathcal{X}^{\sigma}_1$ so that [[I'm confused here by the notation $(Q^{\mathcal{E}}_{\lambda}{\restriction}\eta + 1)$. Does this really mean completely ignore $\vec\nu^q{\lfloor}\eta + 1$? How does it relate to $Q|\eta + 1$?]]

$$\Vdash_{\mathcal{R}_{\lambda}} \left(q' \Vdash \sigma \text{ or } \forall q'' \le^* q'\ (q'' \nVdash \sigma)\right) \quad \text{and} \quad \Vdash_{\mathcal{R}_{\nu+1}} p' \,\|_{\mathcal{R}_{\nu+1,\lambda}}\, q' \Vdash_{Q^{\mathcal{E}}_{\lambda}} \sigma.$$

Now set $g^q_{\gamma}(\nu) = p' {}^\frown \langle \lambda, q' \rangle \in \mathcal{R}^{\mathcal{E}{\restriction}\gamma}_{\nu+1,\lambda+1}$, and define $A^{\nu+1} = A^{q'}_{\gamma}$ and $g^{\nu+1} = g^{q'}_{\gamma}$. $\square$

In the remainder of this subsection we apply the proof of the Prikry Property in [7, lemma 4.15], using Claim 3.60 to permit us to ignore all but the final coordinate $q_\gamma = (\nu^q_\gamma, \beta^q_\gamma, A^q_\gamma, g^q_\gamma, f^q_\gamma)$ of the conditions $q \in Q^{\mathcal{E}}_\lambda$. We need to verify that the proof of [7, lemma 4.15] works for the present forcing. We do not repeat here the proofs from [7], but simply state the appropriate adaptations of the major lemmas from that paper and indicate necessary changes to the notation and proof.

First, if $q \in Q^{\mathcal{E}}_\lambda$ and $\alpha \in A^q_\eta$ then we write $r^q(\alpha)$ for the condition

$$\langle \alpha, \mathrm{pure}(q{\downarrow}\alpha)\rangle^\frown g^q_\gamma(\alpha)^\frown \langle \lambda, \mathrm{add}(q,\gamma,\alpha)\rangle \in \mathcal{R}^{\mathcal{E}}_{\alpha,\lambda+1}.$$

Thus $r^q(\alpha)$ is the weakest condition extending $\langle \lambda, q\rangle$ which forces that $\alpha \in \overline{C}_{\lambda,\gamma,\nu^q_\gamma,f^q_\gamma}$ and that $\bar{o}^{\mathcal{E}}(\xi) < \bar{o}^{\mathcal{E}}(\alpha)$ for all $\xi < \alpha$ in $\overline{C}_{\lambda,\gamma,\nu^q_\gamma,f^q_\gamma}$.

Lemma 3.61 (lemma 4.14 of [7])**.** *Suppose that $\beta < \bar{o}^{\mathcal{E}}(\lambda)$, $p \in \mathcal{R}_\lambda$ and $q \in Q^{\mathcal{E}}_\lambda$, and that D_α is a dense open subset of $(\mathcal{R}^{\mathcal{E}}_{\lambda+1}, \leq^*)$ for each $\alpha \in O_\beta := \{\, \alpha < \lambda : \bar{o}^{\mathcal{E}}(\alpha) = \beta \,\}$. Then there are conditions $p' \leq^* p$ and $q' \leq^* q$ such that for all $\alpha \in A^{q'}_\eta \cap O_\beta$ we have $p'^\frown r^{q'}(\alpha) \leq p^\frown\langle \lambda, q'\rangle$ and $p'^\frown r^{q'}(\alpha) \in D_\alpha$.* $\square$

We are now ready to prove the Prikry property for $Q^{\mathcal{E}}_\lambda$:

Lemma 3.62 (lemma 4.15 of [7])**.** *If $\mathcal{E}$ has successor length $\eta = \gamma + 1$, and $\mathcal{E}$ is semi-complete but not complete, then for each $q \in Q^{\mathcal{E}}_\lambda$ and formula σ there is $q' \leq^* q$ such that $\Vdash_{\mathcal{R}_\lambda} q' \Vdash_{Q^{\mathcal{E}}_\lambda} \sigma$.*

Lemma 3.63. *Let B be the set of $\nu < \lambda$ such that is some $p_\nu \in \mathcal{R}_{\nu+1}$ such that $p \Vdash g^q_\gamma(\nu) \Vdash \mathrm{add}(q,\gamma,\nu) \Vdash \sigma$. If there is $\xi < \bar{o}^{\mathcal{E}}(\lambda)$ such that $B \in \bar{U}^{\mathcal{E}}(\xi)$ then there are $p \in \mathcal{R}_\lambda$ and $q' \leq^* q$ such that $p \Vdash q' \Vdash \sigma$.*

Proof. Write $\bar{U} = \bar{U}^{\mathcal{E}}(\xi)$. For each $\nu \in B$, fix p_ν as specified. There is a set $B' \subseteq B$ in $\bar{U}^{\mathcal{E}}(\xi)$ such that if $\nu < \nu'$ are in B' then $p_\nu = p_{\nu'}{\downarrow}\nu$. Set $q'' = [\nu \mapsto (p_\nu)_\nu]_{\bar{U}}$. Now define q' by letting $A^{q'}{\restriction}\gamma = q''{\restriction}\gamma$, and $\nu \in A^{q'}_\gamma$ if and only if (setting $\bar{\xi} = \bar{o}^{\mathcal{E}}(\nu)$) $\bar{\xi} \propto_{\bar{o}^{\mathcal{E}}(\lambda)}$, $\xi \propto_{\bar{o}^{\mathcal{E}}(\lambda)}$, and $\nu \in A^{q''}_\gamma$ if $\bar{\xi} < \xi$, $\nu \in B'$ if $\bar{\xi} = \xi$, and $\nu \in A^q_\gamma$ if $\bar{\xi} > \xi$.

Set $p' = p{\restriction}\nu$ for all $\nu \in B'$. Then we have the following situation: If ν is any member of B' then $p' \Vdash q'(\nu) \Vdash g^{q'}_\gamma(\nu) \Vdash \mathrm{add}(q,\gamma,\nu) \Vdash \sigma$. Here $q'(\nu)$ is obtained by taking $q'{\downarrow}\nu$ and taking $A^{q'(\nu)}_{\gamma\downarrow\nu}$ equal to the set of $\nu' \in A^{q'}_\gamma \cap \nu$ such that $\bar{o}^{\mathcal{E}}(\nu) < \xi$.

Now it is clear that any $p'' \leq p'$ will satisfy the same property, as will any $q'' \leq^* q'$.

Now suppose that $\xi' > \xi$ and $\nu' \in A^{q'}_\gamma$ with $\bar{o}^{\mathcal{E}}(\nu') = \xi' > \xi$. Then the conditions p' and $q'{\downarrow}\nu'$ will satisfy the same property, with q' replaced by $q'{\downarrow}\nu'$ and ξ replaced by $\xi{\downarrow}\nu'$, because the order of the two extensions could be reversed.

Finally consider the case of ν' with $\xi' = \bar{o}^{\mathcal{E}}(\nu') < \xi$. We consider three cases. Any extension using $\mathrm{add}(q'{\downarrow}\nu', \gamma{\downarrow}\nu', \nu'')$ with $\nu'' \in A^{q'} \cap \nu'$ and $\bar{o}^{\mathcal{E}}(\nu') > \xi$ will inherit the the property as in the last section, as the order of the extensions could have been reversed. Any such extension with $\bar{o}^{\mathcal{E}}(\nu'') = \xi$ will force σ. If

no such extension has been made then we can replace $q'{\downarrow}\nu'$ with q'' obtained by setting $A^{q''}_{\gamma\downarrow\nu'} = \left\{ \nu'' \in A^{q'}_{\gamma} \cap \nu' : \bar{o}^{\mathcal{E}}(\nu'') < \xi \right\}$. The resulting condition will satisfy the property with the original q', as the order of the extensions could have been reversed. $\square$

Proof of lemma 3.62. Let q be arbitrary, and write $q_{*\vec{\nu}}$ for the condition obtained from q by replacing $\vec{\nu}^q$ with $\vec{\nu}$. By taking a $\leq^*$ extension of q we can assume that whenever $\vec{\nu} \propto_{\nu} \bar{o}^{\mathcal{E}}(\lambda)$ the condition $q_{*\vec{\nu}}$ is in $\mathcal{X}^{\sigma}_{2} \cap \mathcal{X}^{\neg\sigma}_{2}$ and, in addition, if $\nu < \nu'$ are in A^q_γ and $\bar{o}^{\mathcal{E}}(\nu) = \bar{o}^{\mathcal{E}}(\nu)$ then ν and ν' agree on whether they satisfy the hypothesis of lemma 3.63.

Now take $q' \leq q$ so that $q' \parallel \sigma$ and $\nu^{q'}_{\gamma}$ is as small as possible. Then $q' \leq q'' \leq q$ where $q' \leq^* \operatorname{add}(q'', \gamma, \nu)$ for some ν. By the choice of q the same would be true of any $\nu' > \nu$ in A^q_γ, and by lemma 3.63 it follows that there is $q''' \leq^* q''$ and $p' \leq p$ which decides σ. This contradicts the minimality of $\nu^{q'}_{\gamma}$ and hence completes the proof of lemma 3.62. $\square$

This is close, but needs work. First, give a name to the property in the hypothesis of lemma 3.63. Then, tighten things up. One thing to check up on. Are things defined so that $(\operatorname{add}(q,\gamma,\nu))_\gamma \leq^* q_\gamma$? I think that this should be true. (But does it help, since the same is not true for $q{\restriction}\gamma$?)

Once lemma 3.62 is proved, we easily have suitability:

Lemma 3.64. *If* $\operatorname{len}(\mathcal{E}) = \gamma + 1$ *and* $\mathcal{E}$ *is semi-complete, but not complete, then* $Q^{\mathcal{E}}_{\lambda}$ *is suitable.*

Proof. Clauses 1–3 and 5 of the definition 2.2 are straightforward, and the remaining clause 4 is the Prikry property, lemma 3.62. $\square$

Similarly, the proofs of lemmas 4.21 and 4.23 of [7] show that CHECK THIS

Lemma 3.65. *If* $\operatorname{len}(\mathcal{E}) = \gamma + 1$ *and* $\mathcal{E}$ *is semi-complete, but not complete, then* $Q^{\mathcal{E}}_{\lambda}$ *is laudable and extends* $\vec{C}(H_\lambda)$. $\square$

3.6 Suitability for complete $\mathcal{E}$ of successor length

1. Enumerate the pairs $(p, \nu) \in \mathcal{R}_\lambda \times \lambda$ as $(p_\iota, \nu_\iota)_{\iota<\lambda}$ so that $\nu_\iota < \iota$ and $p_\iota \in \mathcal{R}_\iota$.

2. (assume for this sketch $\beta^q_\eta = 0$). Do for each $\iota < \lambda$: for each ξ with $\bar{o}^{\mathcal{E}}(\xi) = \iota < \lambda$, choose (if possible) a ordinal $f(\xi) \geq \xi$ and condition $g(\xi) \in \mathcal{R}_\lambda^{\mathcal{E}\upharpoonright\eta}$ so that $g(\xi)\upharpoonright\xi = p$ and $g(\xi)_\xi \leq^* \operatorname{add}(q, \eta, \nu)\downarrow\xi$, and $g(\xi) \Vdash \xi \prec f(\xi)$ & $f(\xi) \in D$. (Probably instead of q here there is a $\leq^*$ sequence of conditions in $Q_\lambda^{\mathcal{E}\upharpoonright\iota}$.)

Shrink $A^q_\eta(\iota)$ so that it is homogeneous for whether this is possible.

If this is not possible,then $f(\xi) = \xi$ and $g(\xi) = q|\iota\downarrow\xi$.

3. Now take a type 3 extension of the resulting condition; say $\lambda' \in Q^{\mathcal{E}}_\lambda$. I claim that this forces $\lambda' \in D$. To see this, let $p' \leq p$ be such that $p' \Vdash \xi \prec \lambda'$ and $\xi \in D$ for some $\xi \geq \lambda'$. Let ι be such that $p_\iota = p'\upharpoonright\lambda'$ and $p'_{\lambda'} \leq^* \operatorname{add}(q\downarrow\lambda', \bar{\eta}, \nu_\iota)$. Then p' witnesses that ξ is a possible choice for $f(\nu)$ where ν is the least member of $C_{\lambda,\iota}$.

Just as in the case of incomplete sequences $\mathcal{E}$, the proofs for the complete sequences only require combining the arguments of [7] with those of section 3.4. First we define our version of Gitik's set D:

The next lemma corresponds to the observation in the proof of theorem 1.1 of Gitik [1], asserting that his forcing $P[E]$ is distributive. The hypothesis is, of course, implied by the assumption that $\mathcal{E}$ is complete, but as in Gitik's theorem we allow the possibility that $\bar{o}^{\mathcal{E}}(\lambda) = 0$, so that $\mathcal{E}$ is not semicomplete.

Lemma 3.66. *Suppose that* $\operatorname{len}(\mathcal{E}) = \gamma + 1$ *and* $\mathcal{E}\upharpoonright\gamma$ *is complete, and that for each* $\beta < \lambda$ *the set*

$$X_\beta = \{\, \nu < \lambda : \mathcal{E}\downarrow\nu \text{ is defined and semicomplete and } \bar{o}^{\mathcal{E}\downarrow\nu}(\nu) = \beta < \nu \,\}$$

is stationary. Then $P^{\mathcal{E}}(\vec{\nu})$ *is* $<\lambda$*-distributive.*

Indeed, if $\{\, D_\alpha : \alpha < \tau \,\}$ *is a set of* $\tau < \lambda$ *open dense subsets of* $P^{\mathcal{E}}(\vec{\nu})$ *then for any* $\nu \in P^{\mathcal{E}}(\vec{\nu})$ *there is a cardinal* $\xi > \nu$ *and a condition* $p \in \mathcal{R}_\lambda$ *with* $\operatorname{domain}(p) = \{\xi\}$ *such that* $p \Vdash (\xi \preccurlyeq \nu \text{ and } \xi \in \bigcap_{\alpha<\tau} D_\alpha)$.

Proof. For $\alpha < \tau$, $\nu < \lambda$ and $p \in \mathcal{R}_\nu$ define $\mu(\alpha, \nu, p)$ to be the least ordinal $\mu > \nu$ such that $p' \Vdash_{\mathcal{R}_\lambda} \mu \in \dot{D}_\alpha$ for some $p' \leq p$ with $p'\upharpoonright\nu + 1 = p$. If $\mu(\alpha, \nu, p)$ is defined, then set $\ell(\alpha, \nu, p) = p'|\nu$ for some such condition p'.

I claim that for each α there is a dense set Y_α of conditions $p \in \mathcal{R}_\lambda$ such that $\mu(\alpha, p, \nu)$ is defined for all $\nu < \lambda$ such that $\sup\operatorname{domain}(p) < \nu$. To see this, fix $\alpha < \tau$ and $p \in \mathcal{R}_\lambda$ and for each $\nu < \lambda$ choose $p_\nu \leq p$ in $\mathcal{R}_\lambda$ and $\xi_\nu > \nu$ so that $p_\nu \Vdash \xi_\nu \in \dot{D}_\alpha$. There is a stationary set S of ν such that $p_\nu\upharpoonright\nu$ is constant, say $p_\nu\upharpoonright\nu = p'$. Then $\mu(\alpha, p', \nu)$ is defined for all $\nu > \sup\operatorname{domain}(p) + 1$, since $\mu(\alpha, p', \nu) \leq \xi'_\nu$ for any $\nu' \in S \setminus \nu + 1$. Hence $p' \in Y_\alpha$.

The hypothesis implies that λ is Mahlo, and hence $\mathcal{R}_\lambda$ has the λ-chain condition. It follows that there is $\delta < \lambda$ so that Y_α is dense in $\mathcal{R}_\delta$ for all $\alpha < \tau$. Let B be the set of $\xi < \lambda$ such that $\mu(\alpha, p, \nu) < \xi$ and $k(\alpha, p, \nu) \in \mathcal{R}_\xi$ for all $\alpha < \tau$, $p \in Y_\alpha \cap \mathcal{R}_\delta$ and $\nu < \xi$. Then B is closed and unbounded, so there is some $\xi > \delta$

in $B\cap X_\delta$. I will show that there is $q\in Q^{\bar{\mathcal{E}}}_\xi$ such that $\Vdash_{\mathcal{R}_\xi}\Vdash_{Q^{\bar{\mathcal{E}}}_\xi}\nu\in\dot{D}_\alpha$. For this it will be sufficient to arrange that $C_{\xi,\bar\gamma}\cap D_\alpha\neq\emptyset$.

We will only need to consider $q_{\bar\gamma}$, so we set $q|\bar\gamma=$?????

Specify what this is, and fill this paragraph out. — Some value of the t functions.

To define $q_{\bar\gamma}$, first set

$$A^q_{\bar\gamma}=\bigcup\{\,X_\beta\cap(\delta,\xi):\beta{\downarrow}\xi\text{ is defined and }\beta{\downarrow}\xi<\delta\,\}.$$

Now let $\langle\,(\alpha_\iota,p_\iota):\iota<\delta\,\rangle$ enumerate the set of pairs (α,p) such that $\alpha<\tau$ and $p\in Y_\alpha\cap\mathcal{R}_\delta$. Then for $\nu\in A^q_{\bar\gamma}$ we set $f^q_{\bar\gamma}(\nu)=k(\alpha_{\bar\sigma^{\bar{\mathcal{E}}}(\nu)},p_{\bar\sigma^{\bar{\mathcal{E}}}(\nu)},\nu)$ and $g^q_{\bar\gamma}(\nu)=\ell(\alpha_{\bar\sigma^{\bar{\mathcal{E}}}(\nu)},p_{\bar\sigma^{\bar{\mathcal{E}}}(\nu)},\nu)$.

Now suppose that $\alpha<\tau$ and $p\in\mathcal{R}_\lambda$. Take $p'\leq p{\restriction}\delta$ in Y_α and let ι be such that $\alpha_\iota=\alpha$ and $p_\iota=p'$. Then it is forced in $\mathcal{R}_\lambda$ that q forces in $Q^{\bar{\mathcal{E}}}_\xi$ that there is some $q'\in G_\xi$ with $\nu^{q'}_{\bar\gamma}=f^q_{\bar\gamma}(\nu)=k(\alpha,p',\nu)$ and $g^{q'}_{\bar\gamma}(\nu){\restriction}\xi=\ell(\alpha,p',\nu)$ in H_ξ. It follows that $k(\alpha,p',\nu)\in C_{\xi,\bar\gamma}\cap D_\alpha$. □

Perhaps there could be a lemma in the previous proof of the Prikry property which asserts the claim in the last paragraph?

Again, the proof of lemma 3.66 is the same as the proof of [7, theorem 5.3] except for necessary changes in notation. Lemma 3.66 easily implies that $Q^{\mathcal{E}}_\lambda$ has the Prikry property, using the same argument as in [7] with $P_\lambda[\mathcal{E}]$ replacing P_λ:

Lemma 3.67 (lemma 6.3 of [7])**.** *If* $\operatorname{len}(\mathcal{E})=\gamma+1$ *and* $\mathcal{E}$ *is complete then for each condition* $q\in Q^{\mathcal{E}}_\lambda$ *and each formula* σ *of the forcing language there is a condition* $q'\leq^* q$ *such that* $\Vdash_{\mathcal{R}_\lambda} q'\parallel_{Q^{\mathcal{E}}_\lambda}\sigma$. □

Corollary 3.68. *If* $\operatorname{len}(\mathcal{E})=\gamma+1$ *and* $\mathcal{E}$ *is complete then* $Q^{\mathcal{E}}_\lambda$ *is suitable and laudable, and extends the layered tree sequence* $\vec{C}(H_\lambda)$.

Proof. Suitability is straightforward, except for the Prikry property which is lemma 3.67. The proof of laudability is the same as in [7, lemma 6.6]. Finally, it is clear that $P^{\mathcal{E}}(\vec\nu)$ extends $\vec{C}(H_\lambda)$, and since $P^{\mathcal{E}}(\vec\nu)$ is a dense subset of $Q^{\mathcal{E}}_\lambda$ it follows that $Q^{\mathcal{E}}_\lambda$ extends $\vec{C}(H_\lambda)$ as well. □

Proof of laudability?

Lemma 3.69. *The forcing* Q_λ *is suitable and laudable, and extends* $\vec{C}(H_\lambda)$.

Proof. This lemma has been proved, with $Q^{\mathcal{E}}_\lambda$ in place of Q_λ, for each of the three cases: $\operatorname{len}(\mathcal{E})$ a limit ordinal, or $\operatorname{len}(\mathcal{E})$ a successor ordinal and $\mathcal{E}$ either complete or incomplete. The suitability and laudability of Q_λ follows immediately, and the only thing remaining to do in order to prove that Q_λ extends $\vec{C}(H_\lambda)$ is to show that the generic set $G_\lambda\subset Q$ can be recovered from $\vec{C}_\lambda=\vec{C}(G_\lambda)$. The argument for each of the three individual cases described how to construct, given

$\vec{C}_\lambda$ and a condition $(\mathcal{E}, q) \in G_\lambda$, a generic subset $G = G^{\mathcal{E}}(\vec{C}_\lambda)$ of $Q^{\mathcal{E}}$ with the property that $\vec{C}_\lambda = \vec{C}(G)$.

We will show that if $(\mathcal{E}, q)$ is an arbitrary condition in Q_λ then $(\mathcal{E}, q) \in G_\lambda$ if and only if the set $G = G^{\mathcal{E}}(\vec{C}_\lambda)$ is defined and is a generic subset of $Q^{\mathcal{E}}_\lambda$ such that $\vec{C}_\lambda = \vec{C}(G)$ and $q \in G^{\mathcal{E}}(\vec{C}_\lambda)$.

The implication from left to right is immediate, and we prove the implication from right to left by induction on $o^*(\vec{C}_\lambda)$. Thus, suppose that $(\mathcal{E}', q') \in G_\lambda$, and that $(\mathcal{E}, q)$ is a condition in Q_λ such that $(\mathcal{E}', q')$ forces that $G = G^{\mathcal{E}}(\vec{C}_\lambda)$ is defined and is a generic subset of $Q^{\mathcal{E}}_\lambda$ such that $\vec{C}_\lambda = \vec{C}(G)$ and $q \in G^{\mathcal{E}}(\vec{C}_\lambda)$. We will show that $(\mathcal{E}, q') \equiv (\mathcal{E}', q')$ and that $q' \leq q$ in $Q^{\mathcal{E}}_\lambda$. Thus $(\mathcal{E}', q') \Vdash_{Q_\lambda} (\mathcal{E}, q) \in \dot{G}_\lambda$.

We can assume that $\mathcal{E}$ is semi-complete. Since $\vec{C}(G) = \vec{C}_\lambda = \vec{C}(G_\lambda)$ we must have $\operatorname{len}(\mathcal{E}) = o^*(\vec{C}_\lambda) = \operatorname{len}(\mathcal{E}')$.

First suppose that $o^*(\vec{C}_\lambda) = \operatorname{len}(\mathcal{E}) = \operatorname{len}(\mathcal{E}')$ is a limit ordinal. For each $\gamma < \operatorname{len}(\mathcal{E})$, the induction hypothesis implies that $Q^{\mathcal{E}\restriction\gamma}_\lambda$ and $Q^{\mathcal{E}'\restriction\gamma}_\lambda$ agree below $q'|\gamma$, and that $(\mathcal{E}\restriction\gamma, q'|\gamma) \leq (\mathcal{E}\restriction\gamma, q|\gamma)$. In particular $\nu^q_\gamma \leq \nu^{q'}_\gamma$ for all $\gamma < \operatorname{len}(\mathcal{E})$ and $\nu^q_\gamma = \nu^{q'}_\gamma$ for all $\gamma \geq \operatorname{len}(\mathcal{E})$. It follows that $\nu^q_\gamma = \nu^{q'}_\gamma$ for all $\gamma > \max(I^{\nu^{q'}} \cap \operatorname{len}(\mathcal{E}) < \operatorname{len}(\mathcal{E})$. It follows from the definition of $Q^{\mathcal{E}}_\lambda$ for this case that $Q^{\mathcal{E}}_\lambda$ agrees with $Q^{\mathcal{E}'}_\lambda$ below q', and hence $(\mathcal{E}', q') \equiv (\mathcal{E}, q') \leq (\mathcal{E}, q)$.

> This isn't quite right, since $Q^{\mathcal{E}}_\lambda$ is not quite separative: Suppose that q and q' agree except that there is some $\nu \in A^{q'}_\gamma \setminus A^q_\gamma$ which can't actually happen — eg, $\mathcal{E}\downarrow\nu$ is not defined. Then $q \Vdash q' \in G_\lambda$ but $q' \not\leq q$.

Now consider the case that $o^*(\vec{C}_\lambda)$ is a successor ordinal, say $o^*(\vec{C}_\lambda) = \gamma + 1$. By the induction hypothesis we can assume that $\mathcal{E}'\restriction\gamma$ agrees with $\mathcal{E}\restriction\gamma$ below $q'|\gamma$. Thus we only need to consider $\mathcal{E}'_\gamma$ and $\mathcal{E}_\gamma$.

Suppose first that λ is singular in $V[\vec{C}]$. Then neither of $\mathcal{E}$ and $\mathcal{E}'$ are complete, so let $q_\gamma = (\beta, \nu, A, g, f)$ and $q'_\gamma = (\beta', \nu', A', g', f')$. Assume first that $\beta = \bar{o}^{\mathcal{E}}(\lambda)$ and $\beta' = \bar{o}^{\mathcal{E}'}(\lambda)$, so both $C_{\lambda,\nu,f}$ and $C_{\lambda,\nu',f'}$ are Prikry-Magidor sets. This can only happen if f and f' are equal; and it follows that $C_{\lambda,\gamma,\nu',f} = C_{\lambda,\gamma,\nu',f'}$. Thus $\bar{o}^{\mathcal{E}}(\lambda) = \bar{o}^{\mathcal{E}'}(\lambda)$ and $\overline{U}^{\mathcal{E}}(\lambda) = \overline{U}^{\mathcal{E}'}(\lambda)$. For $(\mathcal{E}', q')$ to force all this we must have $(A, g) \supset (A', g')$, and it follows that $Q^{\mathcal{E}}_\lambda$ and $Q^{\mathcal{E}'}_\lambda$ agree below q', as required.

If the assumption $\beta = \bar{o}^{\mathcal{E}}(\lambda)$ and $\beta' = \bar{o}^{\mathcal{E}'}(\lambda)$ does not hold, then there must be $(\mathcal{E}, q'') \leq (\mathcal{E}, q)$ and $(\mathcal{E}', q''') \leq (\mathcal{E}', q')$ which do satisfy $\beta^{q''}_\gamma = \bar{o}^{\mathcal{E}}(\lambda)$ and $\beta^{q'''}_\gamma = \bar{o}^{\mathcal{E}'}(\lambda)$. The arguments above then imply that $\bar{o}^{\mathcal{E}}(\lambda) = \bar{o}^{\mathcal{E}'}(\lambda)$ and $f^{q''}_\gamma = f^{q'''}_\gamma \restriction A^{q''}_\gamma$. In order for $(\mathcal{E}', q')$ to force this situation, we must have $\beta^{q'}_\gamma \geq \beta^q_\gamma$ and $(\mathcal{E}', q') \equiv (\mathcal{E}, q') \leq (\mathcal{E}, q)$.

If λ is regular in $V[\vec{C}]$ then both $\mathcal{E}$ and $\mathcal{E}'$ are complete. Set $\vec{\nu} = \vec{\nu}^{q'}$, so $\vec{C}_\lambda$ is smoothly joined to $\vec{C}$ at $\vec{\nu}$, and consider the dense subsets $P^{\mathcal{E}'}(\vec{\nu}) \subset Q^{\mathcal{E}'}_\lambda$ and $P^{\mathcal{E}}(\vec{\nu}) \subset Q^{\mathcal{E}}_\lambda$. We have $G^{\mathcal{E}'}(\vec{C}_\lambda) \cap P^{\mathcal{E}'}(\vec{\nu}) = C_{\lambda,\gamma} \setminus \nu_\gamma = G^{\mathcal{E}}(\vec{C}_\lambda) \cap P^{\mathcal{E}}(\vec{\nu})$, and in order that q' forces this equality we must have that $P^{\mathcal{E}}(\vec{\nu})$ agrees with $P^{\mathcal{E}'}(\vec{\nu})$

above ν_γ, and hence $(\mathcal{E}', q') \equiv (\mathcal{E}, q') \leq (\mathcal{E}, q)$. $\square$

4 The final model

The last section described the preliminary forcing, adding closed and unbounded subsets of cardinals less than κ. In this section we will first choose a suitable sequence $\mathcal{E}_\kappa$ of length κ^+ and then combine the forcings $Q_\kappa^{\mathcal{E}\restriction\beta}$ in order to add κ^+-many closed unbounded subsets of κ. We will then define the final model as a submodel N of this model, and verify that $N \models \mathrm{ZF} + \mathrm{DC} +$ "$\mathcal{F}_{\mathrm{NS}}$ is an ultrafilter." This will complete the proof of theorem 1.1.

4.1 The forcing at κ

Because the forcing $Q^{\mathcal{E}}$ described in the previous section relies heavily on the assumption that $\operatorname{len}(\mathcal{E}) < \kappa^+$, the forcing at κ must be defined as a special case. The first step is to use the repeat point W to define the sequence $\mathcal{E}_\kappa$ to be used at κ:

Definition 4.1. The sequence $\mathcal{E}_\kappa$ is defined by the following forcing construction, carried out in the ground model: First, fix a transitive model $M^* \prec \mathrm{Hered}_{\kappa^{++}}$ such that $|M^*| = \kappa^+$. Define a partial ordering $(T, \leq^T)$ in M^* as follows: the members of T are the continuously decreasing sequences $\mathcal{E}$ on κ with $\operatorname{len}(\mathcal{E}) < \kappa^+$ such that $E_\iota \in W$ for each term $\mathcal{E}_\iota = (E_\iota, h_\iota)$ in $\mathcal{E}$, and the ordering $\leq^T$ on T is by end extension. Thus $(T, \leq^T)$ is equivalent to the forcing to add a Cohen subset of κ^+.

Since $(T, \leq^T)$ is κ^+-closed and $|M^*| = \kappa^+$, there is a set $G \in V$ which is a M^*-generic subset of $(T, \leq^T)$. Fix such a set G and set $\mathcal{E}_\kappa = \bigcup G$.

The forcing Q^*_κ used at κ is essentially the direct limit of the forcings $Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}$ for $\eta < \kappa^+$. The new sequence $\vec{C}_\kappa$ of closed unbounded subsets of κ obtained from this forcing will be smoothly joined to $\vec{C}$ at $\vec{0}$. This has the undesirable consequence that κ and κ^+ are both collapsed by the forcing, but (at least with the present technology) this seems to be unavoidable. We will simplify some of the earlier notation by using a convention that a missing sequence $\vec{\nu}$ is to be taken to be $\vec{0}$. Thus, for example, we will write $P_\kappa^{\mathcal{E}}$ for $P_\kappa^{\mathcal{E}}(\vec{0})$ and $t^{\mathcal{E}}_{\eta,\nu}$ for $t^{\mathcal{E}}_{\eta,\nu}(\vec{0})$.

The notation $t^{\mathcal{E}}_{\eta,\nu}(\vec{\nu})$ doesn't seem to be defined anywhere. I'm guessing: $t^{\mathcal{E}}_{\gamma,\nu}(\vec{\nu})$ is the weakest condition q in $Q^{\mathcal{E}}_\lambda$ such that $\vec{\nu}^q = \operatorname{add}(\vec{\nu}, \gamma, \nu)$. Here λ is the appropriate cardinal for $\mathcal{E}$, which would be κ in this section.

If I set $t^{\mathcal{E}}(\vec{\nu}) = \langle\, (\nu_\iota, E_\iota, h_\iota, 0, \emptyset, \emptyset) : \xi < \kappa^+ \,\rangle$ then $t^{\mathcal{E}}_{\gamma,\nu}(\vec{\nu})$ would be $\operatorname{add}(t^{\mathcal{E}}(\vec{\nu}), \gamma, \nu)$.

I'm not sure that $t^{\mathcal{E}}_0$ is defined either. This would be $t^{\mathcal{E}}_0(\vec{\nu}) = t^{\mathcal{E}}(\vec{\nu}) = \langle\, (\nu_\iota, E_\iota, h_\iota, 0, \emptyset, \emptyset) : \iota < \kappa^+ \,\rangle$.

Should γ be a successor ordinal here? Maybe not. The point is that (γ, ν) will force that $\nu = \min(C_{\kappa,\gamma})$.

Definition 4.2. The members of the forcing Q^*_κ are pairs (γ,ν) such that $\nu \in P^{\mathcal{E}_\kappa\restriction\gamma}$ and $(\mathcal{E}_\kappa\restriction\gamma, t_0^{\mathcal{E}_\kappa\restriction\gamma})\downarrow\nu \in G_\nu$.

If (γ',ν') and (γ,ν) are in Q^*_κ then $(\gamma',\nu') \le (\gamma,\nu)$ if and only if $\gamma \propto_{\nu'} \gamma'$ and $(\mathcal{E}_\kappa\restriction\gamma', t_{\gamma,\nu}^{\mathcal{E}_\kappa\restriction\gamma'})\downarrow\nu' \in G_{\nu'}$.

Definition 4.3. If $G \subset Q^*_\kappa$ is $V[H_\kappa]$-generic then we write $\vec{C}(G)$ for the sequence defined by setting $C_{\kappa,\gamma} = \bigcup\{\, C_{\nu,\gamma\downarrow\nu} : \exists\gamma'\big((\gamma',\nu) \in G \text{ and } \gamma \propto_\nu \gamma'\big)\,\}$ for each $\gamma < \kappa^+$.

$\boxed{\text{Actually, below I'd want } C_{\nu,\gamma'} = \emptyset \text{ for all } \gamma' \ge \gamma\downarrow\nu.}$

The forcing Q^*_κ could equivalently be defined by $(\gamma,\nu) \in Q^*$ if $o^*(\nu) = \gamma\downarrow\nu$, $C_{\nu,\gamma\downarrow\nu} = \varnothing$, and $\mathcal{E}_\nu = (\mathcal{E}_\kappa\restriction\gamma)\downarrow\nu$ is complete. Notice that (γ,ν) forces that $\nu = \min(C_{\kappa,\gamma})$ and $C_{\kappa,\gamma'} \cap (\nu+1) = \varnothing$ for all $\gamma' > \gamma$. It follows that forcing with Q^*_κ collapses κ^+, since the function mapping $\gamma < \kappa^+$ to $\min\bigcup_{\gamma'\ge\gamma} C_{\kappa,\gamma'}$ is nondecreasing and is cofinal in κ. Indeed, κ^+ is collapsed onto ω: let $\langle\, A_\gamma : \gamma < \kappa^+ \,\rangle$ be any sequence of sets such that $A_\gamma \in \mathcal{U}^{\mathcal{E}_\kappa}(\kappa, \kappa^+\cdot\gamma)$ and $|E_\kappa, \gamma \setminus A_\gamma| = \kappa$. Then $x = \{\, \gamma : \min(C_{\kappa,\gamma}) \notin A_\gamma \,\}$ is unbounded in κ^+ but has no limit points below κ^+, so that $\{\, \min(C_{\kappa,\gamma}) : \gamma \in x \,\}$ is a cofinal subset of κ of order-type ω.

Proposition 4.4. *If $G \subset Q^*_\kappa$ is generic and $\vec{C} = \vec{C}(G)$ then $\vec{C}\restriction\eta{+}1$ is $Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}$-generic for any $\eta < \kappa^+$.*

Proof. Let $D \subset Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}$ be dense, and let (η',ν') be any condition in Q^*. Pick $\eta'' > \max(\eta,\eta')$. Then we can pick ν'' so that $(\eta'',\nu'') \le (\eta',\nu)$ and $D\downarrow\nu''$ is dense in $Q_{\nu''}^{(\mathcal{E}\restriction\eta+1)\downarrow\nu''}$. Then (η'',ν'') forces that there is some condition in $G_{\nu''}^{(\mathcal{E}\restriction\eta+1)\downarrow\nu''} \cap D \subset G_\kappa^{\mathcal{E}\restriction\eta+1}$. □

As a corollary we get that Q^*_κ can be factored as $Q^*_\kappa \equiv Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1} * \dot{S}_{\kappa,\eta}$ for some partial order $S_{\kappa,\eta}$. The next proposition gives an explicit construction of $S_{\kappa,\eta}$:

Proposition 4.5. *Suppose $(\eta,\nu_0) \in Q^*$. Then*

$$Q^*/(\eta,\nu_0) \equiv Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}/t_{\eta,\nu_0}^{\mathcal{E}_\kappa\restriction\eta+1} * \dot{S}_{\kappa,\eta}$$

where if $G \subset Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}$ is $V[H_\kappa]$-generic, with $t_{\eta,\nu_0}^{\mathcal{E}_\kappa\restriction\eta+1} \in G$, then $S_{\kappa,\eta}$ is defined in $V[H_\kappa, G]$ to be the subordering of Q^*_κ consisting of those pairs $(\gamma,\nu) \in Q^*_\kappa$ such that $\eta \propto_\nu \gamma$ and $t_{\eta,\nu}^{\mathcal{E}_\kappa\restriction\eta+1} \in G$.

Equivalently, $(\gamma,\nu) \in S_{\kappa,\eta}$ if $(\gamma,\nu) \in Q^*_\kappa$, $\eta \propto_\nu \gamma$ and $\nu \in C_{\kappa,\eta}$.

Proof. Define a function σ by $\sigma(\gamma,\nu) = (t_{\eta,\nu}^{\mathcal{E}_\kappa\restriction\eta+1}, (\gamma,\nu))$ whenever $\eta \propto_\nu \gamma$. Then σ is an order preserving map from $Q^*_\kappa/(\eta,\nu_0)$ into $Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}/t_{\eta,\nu_0}^{\mathcal{E}_\kappa\restriction\eta+1} * \dot{S}_{\kappa,\eta}$.

To see that range(σ) is dense, let $(q,(\gamma,\nu)) \in Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}/t_{\eta,\nu_0}^{\mathcal{E}_\kappa\restriction\eta+1} * \dot{S}_{\kappa,\eta}$ be arbitrary. Then $q < t_{\eta,\nu}^{\mathcal{E}_\kappa\restriction\eta+1}$ in $Q_\kappa^{\mathcal{E}_\kappa\restriction\eta+1}$ since $q \Vdash t_{\eta,\nu}^{\mathcal{E}_\kappa\restriction\eta+1} \in \dot{G}_\kappa$.

Take $(\gamma+1,\nu')$ with $t_{\eta,\nu'}^{\mathcal{E}_\kappa\restriction\eta+1} \le q$ and $(\gamma+1,\nu') \le (\gamma,\nu)$. Then $\sigma(\gamma+1,\nu') \le (q,(\gamma,\nu))$.

To see that there is such a ν', notice that $t_0^{\mathcal{E}_\kappa\restriction\gamma}(\vec{\nu}^q)$ forces that $(i^U(\gamma+1),\kappa)$ is such a condition in $\mathrm{ult}(V,U)$ where $U = \mathcal{U}(\kappa, \gamma\cdot\kappa^+)$. □

4.2 The submodel

For the remainder of the paper fix a V-generic subset $H_{\kappa+1}$ of $\mathcal{R}_{\kappa+1}$ and a $V[H_{\kappa+1}]$-generic subset K of the Levy collapse $\mathrm{Levy}(\kappa,\omega)$, so that $V[K]\models\kappa=\omega_1$, and $M=V[H_{\kappa+1},K]$. Our final model will be a submodel N of M.

Definition 4.6. If $\gamma<\kappa^+$ then we write M_γ for $V[H_{\kappa+1}|\gamma,K]=V[H_\kappa,\vec{C}_\kappa\restriction\gamma,K]$.

Thus $M_0=V[H_\kappa,K]$ and $M_{\kappa^+}=M$. If γ is a limit ordinal less than κ^+ then $M_\gamma=M_{\gamma+1}$, since $C_{\kappa,\gamma}$ is (except for an initial segment) equal to $\Delta_{\iota<\gamma}C_{\kappa,\iota}$. Proposition 4.4 implies that $M_{\gamma+1}$ is a generic extension of V by the forcing $\mathcal{R}^{\mathcal{E}_\kappa\restriction\gamma+1}_{\kappa+1}\times\mathrm{Levy}(\kappa,\omega)$.

Notice that $M_{\gamma+1}$ satisfies the axiom of choice, and that $\omega_1^{M_{\gamma+1}}=\kappa$ and $\omega_2^{M_{\gamma+1}}=(\kappa^+)^V$.

Definition 4.7. We write $\mathcal{C}$ for the set of sequences $\vec{D}$ in M such that

1. $d=\mathrm{domain}(\vec{D})\in\mathcal{P}^V_\kappa(\kappa^+)$, and d is closed in κ^+.
2. There is some $\nu<\kappa$ so that $D_\kappa\setminus\nu=C_{\kappa,\gamma}\setminus\nu$ for all $\gamma\in d$, and $\langle\, D_\gamma\cap\nu:\gamma\in d\,\rangle\in M_0$.

If $d\in\mathcal{P}^V_\kappa(\kappa^+)$ is closed then we write $\vec{C}_{\kappa,d}$ for the sequence $\langle\, C_{\kappa,\gamma}:\gamma\in d\,\rangle\in\mathcal{C}$.

We are now ready to define the final model. The following theorem completes the proof of the main theorem 1.1, and its proof will take up the rest of this paper.

Instead of $\mathrm{Hered}^{M_0}_{\omega_1}$ I should probably just use $\mathcal{P}^{M_0}(\omega)$. The two are equivalent, since M_0 satisfies the AC.

Theorem 4.8. *Let N be the model*

$$N=\mathrm{HOD}^M(V\cup\mathrm{Hered}^{M_0}_{\omega_1}\cup\mathcal{C}\cup\{\mathrm{Hered}^{M_0}_{\omega_1},\mathcal{C}\}).$$

Then every cardinal $\lambda\geq\kappa$ is a cardinal in N, and N satisfies ZF + DC + $\kappa=\omega_1^N$ + "the filter of closed unbounded subsets of κ is an ultrafilter."

The following lemma states a homogeneity property which will be sufficient for all remaining proofs except for the proof that the filter of closed, unbounded sets is an ultrafilter.

7/28/07 — At least with the current proof I don't think that this old comment is correct:

It may be of interest to note that the proof of lemma 4.9 is not dependent on the use of the Levy collapse $\mathrm{Levy}(\kappa,\omega)$: The collapse $\mathrm{Levy}(\kappa,\lambda)$ to make $\kappa=\lambda^+$ would work as well, for any cardinal λ less than the first measurable cardinal.

Lemma 4.9. *Assume that $H\times K$ is a generic subset of $\mathcal{R}_{\kappa+1}\times\mathrm{Levy}(\kappa,\omega)$, that $p_0\in H^{\mathcal{E}_\kappa\restriction\gamma+1}_{\kappa+1}$, that $p_0\Vdash(\iota,\alpha)\in\dot{S}_{\kappa,\gamma}$, and that $\lambda_0<\kappa$. Then there is a generic set $H'\times K'$ such that $(\iota,\alpha)\in G'_\kappa$ (where $H'=H'_\kappa*G'_\kappa$), and $H'\times K'$ agrees with $H\times K$ as follows:*

1. $V[H', K'] = V[H, K]$.
2. $H'_{\lambda_0} = H_{\lambda_0}$ *and* $K'_{\lambda_0} = K_{\lambda_0}$.
3. $p_0 \in {H'}^{\mathcal{E}_\kappa \restriction \gamma+1}_{\kappa+1}$.
4. $\vec{C}^{H'}_\kappa \restriction \gamma = \vec{C}_\kappa \restriction \gamma$.
5. $\mathcal{C}^{H',K'} = \mathcal{C}$.
6. $\mathcal{P}(\omega)^{M'_0} = \mathcal{P}(\omega)^{M_0}$ *where* $M'_0 = V[H' \restriction \kappa, K'] = (M_0)^{V[H',K']}$.

Proof. Fix some $(\eta, \alpha') \in G$ such that $p \restriction \kappa \in \mathcal{R}_{\alpha'}$, $\lambda_0 < \alpha'$, $\alpha < \alpha'$, $\gamma, \iota \propto_{\alpha'} \eta$ and $(\eta, \alpha') \prec (\iota, \alpha)$ in $P^{\vec{C}}_\kappa$. We will not alter $H_{\alpha'}$ or $K_{\alpha'}$, or $H_{\alpha'+1,\kappa+1}$ or $K_{\alpha'+1,\kappa}$. (H_κ will be changed at points in $C_{\kappa,0}$ above α', but only to reflect changes at α'.)

We first modify $G_\nu \subseteq Q^{\mathcal{E}_{\alpha'}}_{\alpha'} = Q^{(\mathcal{E}_\kappa \restriction \eta) \downarrow \alpha'}_{\alpha'}$ to $G'_{\alpha'}$, simultaneously changing the generic collapse map from ω onto α' in $K_{\alpha'+1} \setminus K_\alpha$. Let $\sigma \colon \omega \to \alpha'$ be the collapse map. Work in $\bar{M} = V[H^{\mathcal{E}_{\alpha'} \restriction (\gamma+1) \downarrow \alpha'}_{\alpha'}, K_{\alpha'}, \sigma]$. Write S for the forcing such that $\mathcal{R}^{\mathcal{E}_{\alpha'}}_{\alpha'+1} = \mathcal{R}^{\mathcal{E}_{\alpha'} \restriction (\gamma+1) \downarrow \alpha'}_{\alpha'+1} * \dot{S}$. In the model $\bar{M}$, the forcing S is countable, as is the forcing S/q where q is the condition $t^{\mathcal{E}_{\alpha'} \restriction (\gamma+1) \downarrow \alpha'}_{\iota \downarrow \alpha', \alpha}$ adding α to $C_{\alpha', \iota \downarrow \alpha'}$. If $\pi \colon S \to S/q$ is the isomorphism [[Actually of the Boolean Algebras]] then we can take $G'_{\alpha'} = G^{\mathcal{E}_{\alpha'} \restriction (\gamma+1) \downarrow \alpha'}_{\alpha'} * \pi``(G^*)$ where $G^* \subseteq S$.

Now we need to change the sets $C_{\lambda,\nu}$ with $\lambda \in C_{\kappa,0}$. Since $H_{\alpha'}$ is unchanged, this is actually easy: $C'_{\lambda,\nu} = C_{\lambda,\nu}$ if $\alpha' \notin C_{\lambda,\nu}$, and $C'_{\lambda,\nu} = C_{\lambda,\nu} \setminus \alpha' \cup C_{\alpha', \nu \downarrow \alpha'}$ if $\alpha' \in C_{\lambda,\nu}$.

For this, go back to where $\mathcal{R}_\kappa \equiv \mathcal{R}_{\alpha'+1} \times \mathcal{R}_{\alpha'+1,\kappa}$.

7/28/07 — Unless I've made some sort of mistake, all of this next part is obsolete.

Case 1. We define $\vec{C}'_\kappa$ as follows:

$$C'_{\kappa,\xi} = \begin{cases} (C_{\kappa,\iota} \setminus \alpha') \cup \{\alpha\} & \text{if } \xi = \iota \\ C_{\kappa,\xi} \setminus \alpha' & \text{if } \iota < \xi \leq \eta \\ C_{\kappa,\xi} & \text{otherwise.} \end{cases}$$

Note that for $\xi > \eta$ we have $C_{\kappa,\xi} \cap \nu = \varnothing$.

Need to verify that $\vec{C}_\kappa$ is constructible from $\vec{C}'_\kappa$.

Should it be $C'_{\kappa,\xi} = C_{\kappa,\xi} \setminus \alpha' \cup \{\alpha\} \cup C_{\alpha,\xi\downarrow\alpha}$ for $\gamma < \xi \propto_\alpha \iota$?

The sequence $\vec{C} \restriction \kappa ^\frown \vec{C}'_\kappa$ is a layered tree sequence; however $\vec{C}'_\kappa$ is not smoothly joined to $\vec{C}$ at $\vec{0}$, but rather at the sequence $\langle \sup(C'_{\kappa,\zeta} \cap (\alpha+1)) : \zeta < \kappa^{++} \rangle$. We will need to define $\vec{C}'_\lambda$ for $\lambda < \kappa$ in such a way as to repair this defect. In the next two cases, $\vec{C}'_\lambda = \vec{C}_\lambda$.

Case 2. $\vec{C}'_\lambda = \vec{C}_\lambda$ for all $\lambda < \alpha'$. Note that this is required by clause 4.9(2).

Case 3. $\vec{C}'_\lambda = \vec{C}_\lambda$ for all $\lambda \notin C_{\kappa,0}$. There is no need to change $\vec{C}_\lambda$ in this case, since $\vec{C}_\lambda$ need not be smoothly joined to $\vec{C}' \restriction \lambda$ at $\vec{0}$.

For the remaining cases we use a recursion on $\lambda \in C_{\kappa,0} \setminus \alpha'$. The next two cases are determined by the requirement that $\vec{C}'_\kappa$ is joined to $\vec{C}'$ at $\vec{0}$, and the fact that we want $C'_{\lambda,\delta} \setminus \alpha' + 1 = C_{\lambda,\delta} \setminus \alpha' + 1$.

Is this for all δ, or has a δ been specified.

Case 4. If $\lambda \in C_{\kappa,\zeta}$ for some $\zeta < \kappa^+$ then $C'_{\lambda,\zeta\downarrow\lambda} = C'_{\kappa,\zeta} \cap \lambda$.

Case 5. If $C_{\lambda,\zeta} \setminus \alpha' + 1 \neq \varnothing$ then $C'_{\lambda,\zeta} = C'_{\mu,\zeta\downarrow\mu} \cup C_{\lambda,\zeta} \setminus \mu$ for any $\mu \in C_{\lambda,\zeta} \setminus (\alpha' + 1)$.

Some more explaination needed here. At least at first glance, clause 4 seems to cover it all. In the next paragraph, should "1 and 2" be "4 and 5"?

It should be noted the the result of case 5 does not depend on the choice of μ, and that when cases 1 and 2 are both applicable then they give the same result.

So far all of the changes have been determined by the choice of the ordinals ι, α, and α'. Furthermore, the reversal is determined by ι, α, α', and ξ_0: given the portion of $\vec{C}'$ which has been defined so far, the corresponding portion of $\vec{C}$ is determined by these ordinals. The remaining case will not have this property: we will take the information needed to define $\vec{C}'$ from K, and we will encode into K' the information needed for the reversal.

Is $\sup(C_{\lambda,\zeta})$ supposed to be $\sup(C_{\lambda,\zeta} \cap \alpha' + 1)$?? It still doesn't make sense.

Case 6. If none of the cases above hold, then define the sequence $\vec{\lambda}$ by $\lambda_\zeta = \max(C_{\lambda,\zeta} \cap \alpha' + 1)$. Then $\vec{\lambda}$ is a member of P_λ, and $0 < \lambda_\zeta = \sup(C_{\lambda,\zeta})$ for some ζ. We define $\vec{C}'_\lambda$ by setting

$$C'_{\lambda,\zeta} = \begin{cases} (C_{\lambda,\zeta} \setminus \alpha') \cup \{\lambda'_\zeta\} \cup C_{\lambda'_\zeta,\zeta\downarrow\lambda'_\zeta} & \text{if } \lambda'_\zeta > 0 \\ C_{\lambda,\zeta} \setminus \alpha' & \text{if } \lambda'_\zeta = 0 \end{cases}$$

where $\vec{\lambda}'$ is a sequence, to be determined, in P_λ. An initial segment of $\vec{\lambda}'$ has already been determined by cases 1 and 2, and for smoothness we must have $\lambda'_\zeta \in C_{\lambda'_{\zeta'},\zeta'\downarrow\lambda'_{\zeta'}}$ whenever $\zeta' < \zeta$ and $0 < \lambda'_\zeta < \lambda'_{\zeta'}$. [[Is this the only criteria needed to ensure the sequence works?]] There is at least one possible choice for $\vec{\lambda}'$, namely the one obtained by setting $\lambda'_\zeta = 0$ whenever it is not already determined, and there are at most $|P_\lambda| = \lambda^{++}$ possible choices for this sequence. We will use the Levy collapse of λ^{++} to make the choice of $\vec{\lambda}'$, and in the process we will define the interval $K'_{\lambda,\lambda^{++}}$ of K'.

It looks like there are λ^+, not λ^{++}, choices of $\vec{\lambda}'$. Should λ^{++} in the next paragraph be λ^+?

To see how to use this collapse, first note that the forcing $\mathrm{Fn}(\lambda^{++}, \omega)$ collapsing λ^{++} onto ω is isomorphic to $\mathrm{Fn}(\lambda^{++}, \omega) \times \mathrm{Fn}(\omega, 2)$, that is, to the collapse followed by a Cohen real. In going from K to K' we will leave the first term in this product, the collapse, fixed, and we will modify the second term, the Cohen real. Write b_λ and b'_λ for this Cohen real in K and K' respectively. After the collapse of λ^{++} there are at most countably many legal choices for $\vec{\lambda}'$ (and for $\vec{\lambda}$ in going in the reverse direction to define $\vec{C}$ from $\vec{C}'$). Define for each $n \leq \omega$ an isomorphism $\pi_n \colon \mathrm{Fn}(\omega, 2) \cong \mathrm{Fn}(\omega, 2) \times n$, where n is the trivial order on n elements. Now we can finish the definition of $\vec{C}'_\lambda$ and $K'_{\lambda^{++}}$ by defining $\vec{\lambda}'$ and b'_λ: Suppose that there are $n \leq \omega$ choices $\{\, z_k : k < n \,\}$ for $\vec{\lambda}'$, and let $(a, k) = \pi_n(b_\lambda)$. Then $\vec{\lambda}' = z_k$, the kth choice among the n legal sequences. Now suppose that for the reverse direction there are n' possible choices $\{\, z'_k : k < n' \,\}$ for $\vec{\lambda}$, and that $\vec{\lambda} = z'_{k'}$. Then $b'_\lambda = \pi^{-1}_{n'}(a, k')$.

The following has to be adapted to the current construction.

7/28/07 — The part above is obsolete.

This completes the definition of $\vec{C}'$ and of K'. The construction produces a layered tree sequence $\vec{C}'$ such that $\vec{C}'_\kappa$ is smoothly joined to $\vec{C}'$ at $\vec{0}$, and such that $V[\vec{C}, K] = V[\vec{C}', K']$. Next, we need to verify that $\vec{C}'$ and K' are generic.

7/28 — Note: This proof has $K' = K$. And it is obvious that H' is generic. This next lemma seems to be obsolete.

Lemma 4.10. *There are dense subsets X and X' of $\mathcal{R}_{\kappa+1} \times \mathrm{Levy}(\kappa, \omega)$ and an order preserving isomorphism $\tau \colon X \cong X'$, such that the set $\tau``(X \cap (H \times K))$ generates a generic subset $H' \times K'$ of $\mathcal{R}_{\kappa+1} \times \mathrm{Levy}(\kappa, \omega)$, such that K' is the set defined above and $\vec{C}(H')$ is the sequence $\vec{C}'$ defined above.*

Proof. The crucial observation is that $C_{\kappa,0} \cap \mathrm{domain}(p)$ is finite for every $p \in \mathcal{R}_\kappa$: if $C_{\kappa,0} \cap \mathrm{domain}(p)$ were infinite then it would have a limit point $\xi < \kappa$, but this is impossible because each member of $C_{\kappa,0}$ is inaccessible in V and $|\mathrm{domain}(p) \cap \xi| < \xi$.

We take X to be the set of conditions $(p, r) \in \mathcal{R}_{\kappa+1} \times \mathrm{Levy}(\kappa, \omega)$ such that, first, p determines the value of the finite set $\mathrm{domain}(p) \cap C_{\kappa,0}$ and, second, for each ordinal λ in this finite set (p, r) determines all of the data used in case 6 to define $\vec{C}'_\lambda$ and $K'(\lambda^{++})$, namely the sequences $\vec{\lambda}$ and $\vec{\lambda}'$, the number n of possible choices for the sequence $\vec{\lambda}'$ and the index k of the sequence actually choosen, and the similar numbers n' and k' from the reverse direction.

Then X is a dense subset of $\mathcal{R}_{\kappa+1} \times \mathrm{Levy}(\kappa, \omega)$. To see this, first note that for any condition p there is $p' \leq^* p$ with the same domain such that $p' \Vdash \lambda \notin C_{\kappa,0}$ for all $\lambda \in \mathrm{domain}(p)$ such that $p \nVdash \lambda \in C_{\kappa,0}$. Now fix an ordinal λ which is forced to be in $C_{\kappa,0} \cap \mathrm{domain}(p)$. It may be that any condition $p' \leq p$ which forces the values of the other data for λ has members $\lambda' \in \mathrm{domain}(p') \setminus \mathrm{domain}(p)$

which are also forced to be in $C_{\kappa,0}$; however any such ordinals λ' will be smaller than λ. This ensures a condition $p_n \in X$ can be reached by a finite sequence $p_n < \cdots < p_1 < p_0 = p$ such that each condition p_{i+1} determines all of the required data for each $\lambda \in \operatorname{domain}(p_i) \cap C_{\kappa,0}$.

Any condition $(p,r) \in X$ determines a pair (p',r'), obtained by modifying the sequences $\vec{\nu}^{p_\lambda}$ for $\lambda \in \operatorname{domain}(p) \cap (C_{\kappa,0} \cup \{\kappa\})$ as described in the construction of $\vec{C}'$ from $\vec{C}$, and then making the changes to r as described in the definition of K'. The function obtained by setting $\tau(p,r) = (p',r')$ has the desired properties. $\square$

Now $V[H',K'] = V[\vec{C}',K'] = V[\vec{C},\vec{K}] = V[H,K]$, so clause 1 of lemma 4.9 holds. Clause 2 holds since $\alpha' > \lambda_0$, $H'_{\alpha'} = H_{\alpha'}$, and $K'_{\alpha'} = K_{\alpha'}$. The construction gave $\vec{C}_k^{H'} \restriction \gamma+1 = \vec{C}_k \restriction \gamma+1$, which is clause 4, and since $\operatorname{domain}(p_0 \restriction \kappa) \subset \lambda_0$ it follows that $p_0 \in H'^{\mathcal{E}_\kappa \restriction \gamma+1}_{\kappa+1}$, which is clause 3.

6/19/07 — Under the current construction, K'_κ and K_κ are equal except for $G_{\alpha'}$. At the place where they differ, we have $G_{\alpha'} \times K(\alpha')$ are equiconstructible with $G'_\alpha \times K'(\alpha')$ (using only $H_{\alpha'} = H'_{\alpha'}$). The construction of K'_κ also depends on $C_{\kappa,0}$, which also serves to reverse the process. Thus $M_1^{H',K'} = M_1^{H,K}$. It looks like $H'_\kappa \notin M_0^{H,K}$, so $M_0^{H',K'} \neq M_0^{H,K}$; however $\mathcal{P}^{M_0}(\omega) = \mathcal{P}^{M_1}(\omega)$ because of the Prikry property, so $M_0^{H',K'}$ and $M_1^{H,K}$ have the same subsets of ω.

The construction of H' and K' depends on $\vec{C}_\kappa$, and hence $M_0^{H',K'} \neq M_0$. However, the construction only uses $\vec{C}_\kappa \restriction \eta+1$, since $C_{\kappa,\zeta} \cap \nu = \varnothing$ for all $\zeta \geq \eta+1$. Thus, if ν' is any member of $C_{\kappa,\eta+1}$ then $H'_{\nu'} \times K'_{\nu'}$ can be constructed from $H_{\nu'+1} \times K'_{\nu'}$, and hence $(\operatorname{Hered}^{M_0}_{\omega_1})^{H',K'} = \operatorname{Hered}^{M_0}_{\omega_1}$, clause 6 holds. Since $\vec{C}^{H'}_{\kappa,d} \in \mathcal{C}$ for all d, it follows that $\mathcal{C}^{H',K'} = \mathcal{C}$. Thus clause 5 holds, and this completes the proof of lemma 4.9. $\square$

Lemma 4.11. *For every subset X of $V \times \operatorname{Hered}^{M_0}_{\omega_1}$ which is a member of N there is some $\gamma < \kappa^+$ such that $X \in M_{\gamma+1}$.*

Proof. Since $X \in N$, X is definable in M as the unique set x such that

$$M \models \phi(x, \vec{D}, z_0, a_0, \mathcal{C}, \operatorname{Hered}^{M_0}_{\omega_1}) \tag{10}$$

for some formula ϕ and some $\vec{D} \in \mathcal{C}$, $z_0 \in \operatorname{Hered}^{M_0}_{\omega_1}$ and $a_0 \in V$. Set $d = \operatorname{domain}(\vec{D})$. We can assume without loss of generality that $\vec{D} = \vec{C}_{\kappa,d}$, since otherwise if ν is as in definition 4.7 and $z = \langle D_\gamma \cap \nu : \gamma \in d \rangle$ then $z \in \operatorname{Hered}^{M_0}_{\omega_1}$ and X is definable from $\vec{C}_{\kappa,d}$ together with $\langle z_0, z \rangle \in \operatorname{Hered}^{M_0}_{\omega_1}$.

Let $\dot{z}_0$ be a $\mathcal{R}_\kappa \times \operatorname{Levy}(\kappa,\omega)$-name for z_0, and let $\dot{X}$ be a $\mathcal{R}_{\kappa+1} \times \operatorname{Levy}(\kappa,\omega)$-name for X. Let $(p_0, r_0) \in H_{\kappa+1} \times K$ be a condition which forces that $\dot{X}$ is the unique set x satisfying (10). By enlarging d if necessary, we can assume without loss of generality that $p_0 \in \mathcal{R}^{\mathcal{E}_\kappa \restriction \gamma+1}_{\kappa+1}$, where $\gamma = \sup(d)$.

In order to show that $X \in M_{\gamma+1}$, let w be an arbitrary member of $V \times \operatorname{Hered}^{M_0}_{\omega_1}$ and let $\dot{w}$ be a $\mathcal{R}_\kappa \times \operatorname{Levy}(\kappa,\omega)$-name for w. Recall that $\mathcal{R}_{\kappa+1} =$

$\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa\restriction\gamma+1} * \dot{S}_{\kappa,\gamma}$, so that M is a generic extension of $M_{\gamma+1}$ by $S_{\kappa,\gamma}$. It is sufficient to prove that

$$(p_0, r_0) \Vdash_{S_{\kappa,\gamma}} \Big(w \in X \iff M_{\gamma+1} \models \exists(\iota,\alpha) \in S_{\kappa,\gamma}\ (\iota,\alpha) \Vdash w \in \dot{X} \Big). \tag{11}$$

The implication from left to right is immediate, so we will prove the implication from right to left. If $H \subset \mathcal{R}_{\kappa+1}$ is generic then we will write $H = H\restriction\kappa * G_{\kappa,\gamma} * L$ where $G_{\kappa,\gamma}$ is a generic subset of $Q_\gamma^{\mathcal{E}_\kappa\restriction\gamma+1}$ and L is a generic subset of $S_{\kappa,\gamma}$.

Let $p_1 \in H\restriction\kappa * G_{\kappa,\gamma}$ force that $(\iota,\alpha) \in S_{\kappa,\gamma}$, and that $(\iota,\alpha) \Vdash_{\dot{S}_{gk,\gamma}} \dot{w} \in \dot{X}$. Fix λ_0 large enough that $r_0 \in K_{\lambda_0}$, and $\dot{w}$ and $\dot{z}_0$ are $H_{\lambda_0} \times K_{\lambda_0}$-terms. Now let H' and K' be given by lemma 4.9. Since $V[H', K'] = V[H, K]$ and all of the parameters of (10) are the same in the two models, the formula (10) defines the same set $X^{H',K'} = X$. Furthermore, since $p_0 \in H'$, this set is denoted by the same term $\dot{X}$. Clause 2 of lemma 4.9 implies that $\dot{w}^{H',K'} = \dot{w}^{H,K} = w$, and since $(p_1, (\iota,\alpha)) \in H'$ it follows that $\dot{w}^{H',K'} \in \dot{X}^{H',K'}$. Thus $w \in X$. $\square$

Corollary 4.12. *$\omega_1^N = \kappa$, $\omega_2^N = \kappa^{+V}$, and both ω_1^N and ω_2^N are regular in N.*

Proof. Suppose that the corollary is false, and let f be a function in N witnessing this failure. Then f maps ordinals to ordinals, and hence $f \subset V$, so lemma 4.11 implies that $f \in M_{\gamma+1}$ for some $\gamma < \kappa^+$. This is impossible, since $\omega_1^{M_{\gamma+1}} = \kappa$, $\kappa^{+M_{\gamma+1}} = \kappa^{+V}$, and $M_{\gamma+1}$ satisfies the axiom of choice. $\square$

Lemma 4.13. *$N \models DC$.*

Proof. Let $\mathbf{S} \in N$ be a binary relation such that $\forall x \in \mathrm{field}(\mathbf{S})\ \exists y\ x\ \mathbf{S}\ y$. We will define an infinite $\mathbf{S}$-chain which is a member of N. Let $\dot{\mathbf{S}}$ be a name for $\mathbf{S}$, and fix an ordinal α large enough that it is forced that $\mathbf{S} \in V_\alpha^M$ and every member of field($\mathbf{S}$) is definable in V_α^M from parameters in $V_\alpha \cup \mathcal{C} \cup \mathrm{Hered}_{\omega_1}^{M_0} \cup \{\mathcal{C}, \mathrm{Hered}_{\omega_1}^{M_0}\}$:

Definition 4.14. Define a function τ as follows: The domain of τ is a subset of $\omega \times V_\alpha \times \mathcal{C} \times \mathrm{Hered}_{\omega_1}^{M_0}$. Suppose $n < \omega$, $a \in V_\alpha$, $\vec{D} \in \mathcal{C}$, and $b \in \mathrm{Hered}_{\omega_1}^{M_0}$, and let ϕ_n be the formula with Gödel number n. Then $\tau(n, a, \vec{D}, b) = z$ whenever

$$V_\alpha^M \models \forall z' \Big(\phi_n(b, \vec{D}, a, \mathcal{C}, \mathrm{Hered}_{\omega_1}^{M_0}, z') \iff z' = z \Big). \tag{12}$$

If there is no $z \in V_\alpha^M$ satisfying (12) then $\tau(n, a, \vec{D}, b)$ is undefined.

The function τ is definable in M using the parameters $\{V_\alpha, \mathcal{C}, \mathrm{Hered}_{\omega_1}^{M_0}\}$, and hence $\tau \in N$. We use τ to define a relation $\mathbf{S}^*$ in V. Suppose that $\mathbf{S}$ is definable in M from parameters $\vec{C}_{\kappa,d_0} \in \mathcal{C}$ and $z_0 \in \mathrm{Hered}_{\omega_1}^{M_0}$ together with $\mathcal{C}$, $\mathrm{Hered}_{\omega_1}^{M_0}$ and a member of V. Let $\lambda_0 < \kappa$ be large enough that $z_0 \in V[H_{\lambda_0}, K_{\lambda_0}]$, and set $\gamma_0 = \sup d_0$.

Definition 4.15. The members of field($\mathbf{S}^*$) are the 7-tuples $(p, \gamma, \nu, n, a, d, \dot{b})$ which satisfy the following conditions:

1. $p \in \mathcal{R}_\kappa$ and $(\gamma, \nu) \in Q_\kappa^*$.

2. $d_0 \subset d \in [\kappa^+]^{<\kappa}$ and $\gamma \geq \sup d$
3. $\lambda_0 < \nu < \kappa$.
4. $\dot{b}$ is a $\mathcal{R}_\nu \times \text{Levy}(\nu, \omega)$-term for a member of $\text{Hered}_{\omega_1}^{M_0}$.
5. $(p, (\gamma, \nu)) \Vdash_{\mathcal{R}_{\kappa+1}} \tau(n, a, \vec{C}_{\kappa,d}, \dot{b}) \in \text{field}(\dot{\mathbf{S}})$.

If $s = (p, \gamma, \nu, n, a, d, \dot{b})$ and $s' = (p, \gamma', \nu', n, a', d', \dot{b}')$ are in $\text{field}(\mathbf{S}^*)$ then $s\ \mathbf{S}^*\ s'$ if the following two conditions hold:

1. $(p', (\gamma', \nu')) \leq (p, (\gamma, \nu))$.
2. $(p', (\gamma', \nu')) \Vdash_{\mathcal{R}_{\kappa+1}} \tau(n, a, \vec{C}_{\kappa,d}, \dot{b})\ \dot{\mathbf{S}}\ \tau(n', a', \vec{C}_{\kappa,d'}, \dot{b}')$.

Obviously each $s \in \text{field}(\mathbf{S}^*)$ has a $s' \in \text{field}(\mathbf{S}^*)$ such that $s\ \mathbf{S}^*\ s'$, and hence there is an infinite $\mathbf{S}^*$-chain in V. The following proposition is needed in order to capture such a chain in N:

Proposition 4.16. *For any $s = (p, \gamma, \nu, n, a, d, \dot{b}) \in \text{field}(\mathbf{S}^*)$ there is a set $X_s \subset \mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma+1}$, dense below $(p, (\gamma, \nu))$, such that for any condition $r \in X_s$ there is a set $d_{s,r} \in [\kappa^+]^{<\kappa}$ with the following property:*

Set $\gamma_{s,r} = \sup(d_{s,r})$, and let $X_{s,r}$ be the set of $(p', (\gamma_{s,r}, \nu')) \in \mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma_{s,r}+1} \equiv \mathcal{R}_\kappa^{\mathcal{E}_\kappa \restriction \gamma+1} * \dot{S}_{\kappa,\gamma}^{\mathcal{E}_\kappa \restriction \gamma_{s,r}+1}$ such that there is $s' = (p', \gamma_{s,r}, \nu', n', a', d_{s,r}, \dot{b}') \in \text{field}(\mathbf{S}^*)$ with $s\ \mathbf{S}^*\ s'$. Then $X_{s,r}$ is dense in $\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma_{s,r}+1}$ below r.

Proof. Let $s = (p, \gamma, \nu, n, a, d, \dot{b})$ be an arbitrary member of $\text{field}(\mathbf{S}^*)$. Factor $\mathcal{R}_{\kappa+1} = \mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma+1} * \dot{S}_{\kappa,\gamma}$ and let X_s be the set of $r < p$ in $\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma+1}$ such that there is a pair (γ', ν') and quadruple (n', a', d', b') such that

1. $r \Vdash (\gamma', \nu') \in \dot{S}_{\kappa,\gamma}$.
2. $r \Vdash (\gamma', \nu') \Vdash \tau(n, a, \vec{C}_{\kappa,d}, \dot{b})\ \dot{\mathbf{S}}\ \tau(n', a', \vec{C}_{\kappa,d'}, \dot{b}')$.

Clearly X_s is dense below $(p, (\gamma, \nu))$ in $\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma+1}$. For each $r \in X_s$ pick some such condition (γ', ν') and quadruple (n', a', d', b'). We can assume without loss of generality that $\gamma' = \sup(d')$.

Set $d_{s,r} = d'$ and $\gamma_{s,r} = \gamma'$. We need to show that $X_{s,r}$, defined with this choice of $d_{s,r}$, is dense below r in $\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa \restriction \gamma_{s,r}+1}$. We will do so by showing that r forces that there is a condition $(p'', (\gamma_{s,r}, \nu'')) \in H$ which forces that $\tau(n, a, \vec{C}_{\kappa,d}, \dot{b})\ \dot{\mathbf{S}}\ \tau(n'', a'', \vec{C}_{\kappa,d_{s,r}}, \dot{b}'')$.

By lemma 4.9, since $r \Vdash (\gamma', \nu') \in S_{\kappa,\gamma}^{\mathcal{E}_\kappa \restriction \gamma'+1}$, the condition r forces that there are H', K' in M with $(r, (\gamma', \nu')) \in H'$ such that $V[H', K'] = M$, $\mathcal{C}^{H',K'} = \mathcal{C}$, $(\text{Hered}_{\omega_1}^{M_0})^{H',K'} = \text{Hered}_{\omega_1}^{M_0}$, and $(\vec{C}_{\kappa,d})^{H',K'} = \vec{C}_{\kappa,d}$. It follows that if $\vec{D} = (\vec{C}_{\kappa,d_{s,r}})^{H',K'}$ then $r \Vdash \tau(a, \vec{C}_{\kappa,d}, \dot{b})\ \dot{\mathbf{S}}\ \tau(a', \vec{D}, \dot{b}')$. However, since $\vec{D} \in \mathcal{C}$ there is $\lambda < \kappa$ such that $D_\iota \setminus \lambda = C_{\kappa,\iota} \setminus \lambda$ for all $\iota \in d$. Set $b'' = (b', \langle D_\iota \cap \lambda : \iota \in d \rangle$, and pick n'' so that for any z, $\phi_{n''}(b'', \vec{C}_{\kappa,d}, a', \mathcal{C}, \text{Hered}_{\omega_1}^{M_0}, z)$ holds if and only

if $\phi_{n'}(b', \vec{D}, a', \mathcal{C}, \mathrm{Hered}^{M_0}_{\omega_1}, z)$. Hence there is a condition $(p'', (\gamma_{s,r}, \nu'')) \in H$ which forces that $\tau(n, a, \vec{C}_{\kappa,d}, \dot{b})\ \dot{\mathbf{S}}\ \tau(n'', a'', \vec{C}_{\kappa,d_{s,r}}, \dot{b}'')$. It follows that $s'' = (p'', \gamma_{s,r}, \nu'', n'', a'', d_{s,r}, \dot{b}'') \in \mathrm{field}(\mathbf{S}^*)$, and hence $s\ \mathbf{S}^*\ s''$.

Then $(p'', (\gamma_{s,r}, \nu'')) \in X_{s,r}$. We showed that r forces that there is $(p'', (\gamma_{s,r}, \nu'')) \in X_{s,r} \cap \dot{H}$, and it follows that $X_{s,r}$ is dense below r. □

Proposition 4.17. *For any $s_0 \in \mathrm{field}(\mathbf{S}^*)$ there is a sequence $\langle\, \mathcal{X}_n : n < \omega \,\rangle$ so that*

1. *$s_0 \in \mathcal{X}_0$.*
2. *For all $n < \omega$, $\mathcal{X}_n \subset \mathrm{field}(\mathbf{S}^*)$ and $|\mathcal{X}_n| \le \kappa$.*
3. *For all $n < \omega$ and $s = (\gamma, d, p, \nu, n, a, \dot{b}) \in \mathcal{X}_n$ the set of conditions $(p', (\gamma', \nu'))$ such that there is some $s' = (p', \gamma', \nu', n, a', d', \dot{b}') \in \mathcal{X}_{n+1}$ with $s\ \mathbf{S}^*\ s'$ is dense below $(p, (\gamma, \nu))$.*

Proof. We can set $\mathcal{X}_0 = \{s_0\}$. Now suppose that $\mathcal{X}_n$ has been defined. Let the sets X_s for $s \in \mathcal{X}_n$ and $X_{s,r}$ for $r \in X_s$ be as given by proposition 4.16. Since $\mathcal{R}^{\mathcal{E}_\kappa \restriction \eta+1}_{\kappa+1}$ has a dense subset of size κ for each $\eta < \kappa^+$, we can choose dense subsets $X'_s \subset X_s$ and $X'_{s,r} \subset X_{s,r}$ of cardinality at most κ. Then set

$$\mathcal{X}_{n+1} = \left\{\, s_{s,r,r'} : s \in \mathcal{X}_n \text{ and } r \in X'_s \text{ and } r' \in X'_{s,r} \,\right\}$$

where $s_{s,r,r'}$ is the 7-tuple witnessing that $r' \in X_{s,r}$. Then $\mathcal{X}_{n+1}$ has cardinality at most κ and satisfies the required condition. □

Now let $s_0 \in \mathrm{field}(\mathbf{S})$ be arbitrary and let the sets $\mathcal{X}_n$ be given by proposition 4.17. Set $\gamma = \sup(\bigcup\{\, d_{s,r} : \exists n\, s \in \mathcal{X}_n \text{ and } r \in X_s \,\}) < \kappa^+$. Working in $M_{\gamma+1}$, and recalling that $M_{\gamma+1}$ satisfies the axiom of choice and that $\omega_1^M = \kappa$ and $\omega_2^M = \kappa^+$, use recursion on n to choose elements $s_n = (\gamma_n, d_n, p_n, \nu_n, m_n, a_n, \dot{b}_n) \in \mathcal{X}_n$ so that for each $n \in \omega$ we have $r_n = (p_n, (\gamma_n, \nu_n)) \in H \cap \mathcal{R}^{\mathcal{E}_\kappa \restriction \gamma+1}_{\kappa+1}$ and $s_n\ \mathbf{S}^*\ s_{n+1}$. Thus $\langle\, \tau(m_n, a_n, \vec{C}_{\kappa,d_n}, b_n) : n < \omega \,\rangle$ is an infinite $\mathbf{S}$-chain, and it only remains to show that this sequence is a member of N.

First, the sequences $\langle\, m_n : n < \omega \,\rangle$ and $\langle\, b_n : n < \omega \,\rangle$ are countable sequences of members of $\mathrm{Hered}^{M_0}_{\omega_1} = \mathrm{Hered}^{M_{\gamma+1}}_{\omega_1}$, and hence are members of $\mathrm{Hered}^{M_0}_{\omega_1}$.

Now pick a function $\sigma \in V$ mapping κ onto γ. Since $\langle\, d_n : n < \omega \,\rangle$ is a countable sequence of members of $\mathcal{P}_\kappa(\gamma)$, there is some $\xi < \kappa$ such that $\bigcup_n d_n \subset \sigma``\xi$. Hence $\langle\, d_n : n < \omega \,\rangle$ is definable from $\sigma$ and $\langle\, \sigma^{-1}[d_n] : n < \omega \,\rangle \in \mathrm{Hered}^{M_0}_{\omega_1}$, so $\langle\, d_n : n < \omega \,\rangle \in N$. Also, if we let $d$ be the closure of $\sigma``\xi$, then $d \in \mathcal{P}^V_\kappa(\kappa^+)$, so $\langle\, \vec{C}_{\kappa,d_n} : n < \omega \,\rangle = \langle\, \vec{C}_{\kappa,d} \restriction d_n : n < \omega \,\rangle \in N$.

To see that $\langle\, a_n : n < \omega \,\rangle \in N$, note that $\langle\, a_n : n < \omega \,\rangle$ is a countable subset of $A = \left\{\, a : \exists m\, (p, \gamma, \nu, m, a, d, \dot{b}) \in \mathcal{X}_n \,\right\}$. Since $A \in V$ and $|A| = \kappa$ in V, $\langle\, a_n : n < \omega \,\rangle$ is definable from A together with a member of $\mathrm{Hered}^{M_0}_{\omega_1}$.

Finally, since the sequence $\langle\, \left(\vec{C}_{\kappa,d_n}, m_n, a_n, b_n\right) : n < \omega \,\rangle$ and the function τ are each in N, the $\mathbf{S}$-chain $\langle\, \tau(a_n, d_n, b_n) : n < \omega \,\rangle$ is a member of N. □

Lemma 4.18. *In N, the filter of closed and unbounded subsets of ω_1 is an ultrafilter.*

Proof. Let x be an arbitrary set in $\mathcal{P}(\kappa) \cap N$. By lemma 4.11 there is $\gamma < \kappa^+$ such that $x \in M_{\gamma+1}$. Let $\dot{x}$ be a $\mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa\restriction\gamma+1} \times \mathrm{Levy}(\kappa, \omega)$-name for x. As in the proof of lemma 4.11 pick a formula ϕ and sets $z_0 \in \mathrm{Hered}_{\omega_1}^{M_0}$, $d \in \mathcal{P}_\kappa^V(\kappa^+)$ and $a \in V$ so that x is the unique set such that

$$M \models \phi(x, \vec{C}_{\kappa,d}, z_0, a, \mathcal{C}, \mathrm{Hered}_{\omega_1}^{M_0}). \tag{13}$$

Let $\dot{z}_0$ be a $\mathcal{R}_\kappa \times \mathrm{Levy}(\kappa, \omega)$-name for z_0, and let $p_0 \in \mathcal{R}_{\kappa+1}$ and $s_0 \in \mathrm{Levy}(\omega, \kappa)$ be conditions which force that $\dot{x}$ is the set defined by (13). By increasing γ if necessary we can assume that $d \subset \gamma$ and $p \in \mathcal{R}_{\kappa+1}^{\mathcal{E}_\kappa\restriction\gamma+1}$. Fix $\lambda_0 < \kappa$ large enough that $p_0\restriction\kappa \in \mathcal{R}_{\lambda_0}$, $s_0 \in K_{\lambda_0}$, and z_0 is a $\mathcal{R}_{\lambda_0} \times \mathrm{Levy}(\lambda_0, \omega)$-term.

We need to show that there are $\eta < \kappa^+$ and $\lambda < \kappa$ such that $x \setminus \lambda$ is either contained in or disjoint from $C_{\kappa,\eta}$. To determine the ordinal η, recall that $\mathcal{E}_\kappa$ is defined to be M^*-generic for the forcing which uses initial segments of $\mathcal{E}_\kappa$ as conditions, where M^* is a κ-closed elementary substructure of $\mathrm{Hered}_{\kappa^{++}}$ of size κ^+ which contains all subsets of κ along with all other relevant sets. In particular, the name $\dot{x}$ is a member of M^* since it can be coded as a subset of $\kappa \times (\mathcal{R}_\kappa * \dot{P}[\mathcal{E}_\kappa\restriction\gamma + 1])$, which has cardinality κ since $P[\mathcal{E}_\kappa\restriction\gamma + 1] \subset \kappa$. Let $\mathcal{E}$ be any sequence of successor length $\eta \geq \gamma + 1$ such that $\mathcal{E} \supset \mathcal{E}_\kappa\restriction\gamma + 1$. We will define a pair (E, h) so that if $\mathcal{E}_\kappa\restriction\eta + 1 = \mathcal{E}^\frown\langle\eta, (E, h)\rangle$ then (p_0, s_0) forces that $C_{\kappa,\eta}$ is either almost contained in, or almost disjoint from, $\dot{x}$. Since $\mathcal{E}$ was arbitrary, the M^*-genericity of $\mathcal{E}_\kappa$ will then ensure that there is some η such that $\mathcal{E}_\kappa\restriction\eta + 1$ has this property, and this will complete the proof of lemma 4.18.

A standard homogeneity argument shows that if $(p, s) \leq (p_0, s_0)$ is any condition and $\nu > \lambda_0$ then $(p, s) \Vdash \nu \in \dot{x}$ if and only if $(p, s\restriction\lambda_0) \Vdash \nu \in \dot{x}$. Hence we will ignore the Levy condition s in what follows.

By increasing γ if necessary, we can assume that $\eta = \gamma + 1$ Let $\mathcal{E}_{\kappa,\gamma} = (E_\gamma, h_\gamma)$. Define $h(\nu)$ for each $\nu \in E_\gamma$ by using the Prikry property of $\mathcal{R}_{\nu+1,\kappa+1}^{\mathcal{E}\restriction\gamma+1}$ to pick $h(\nu) \leq^* h_\gamma(\nu)$ so that $\Vdash_{\mathcal{R}_{\nu+1}} h(\nu) \parallel_{\mathcal{R}_{\nu+1,\kappa+1}^{\mathcal{E}\restriction\gamma+1}} \nu \in \dot{x}$. Thus, by setting $\mathcal{E}_{\kappa,\eta} = (E, h)$ for some set $E \subset E_{\eta-1}$ we ensure that for each $\nu \in C_{\kappa,\eta}$ the statement $\nu \in \dot{x}$ is decided by a condition in $\mathcal{R}_{\nu+1}$: for each ν let X_ν^+ be the set of conditions $p \leq p\restriction\lambda_0$ in $\mathcal{R}_{\nu+1}$ such that $p \Vdash_{\mathcal{R}_{\lambda+1}} (\nu \in \dot{C}_{\kappa,\eta} \implies \nu \in \dot{x})$ and let X_ν^- be the set of p such that $p \Vdash_{\mathcal{R}_{\lambda+1}} (\nu \in \dot{C}_{\kappa,\eta} \implies \nu \notin \dot{x})$. Then $X_\nu^+ \cup X_\nu^-$ is dense below $p_0\restriction\lambda_0$.

Fix, for the moment, an arbitrary $\nu \in E_{\kappa,\gamma}$ and set $\bar{\gamma} = \gamma{\downarrow}\nu$ and $\bar{\mathcal{E}} = (\mathcal{E}\restriction\gamma + 1){\downarrow}\nu$. We claim that $\nu \in \dot{x}$ is actually decided by a condition in $\mathcal{R}_{\nu+1}^{\bar{\mathcal{E}}}$. Let $p = p\restriction\nu^\frown\langle\nu, (\mathcal{E}_\nu, q)\rangle$ be any condition in $\mathcal{R}_{\nu+1}$ such that $p \leq p_0\restriction\lambda_0$ and $\mathcal{E}_\nu\restriction\gamma+1 = \bar{\mathcal{E}}$. We will show that p is in X_ν^+ or X_ν^- if and only if $p\restriction\nu^\frown\langle\nu, (\bar{\mathcal{E}}, q\|\bar{\gamma} + 1)\rangle$ is in the same set.

Will $\vec{\nu}\lfloor(\gamma+1)\downarrow\nu$ be a problem?

Maybe this can be simplified, but I think we can say that since $\vec{C}_\kappa$ is smoothly joined at $\vec{0}$, we can assume that $\vec{\nu}^q$ is such that (there is something in H_ν forcing that) $\vec{\nu}^q\|\gamma+1 = (\vec{\nu}^q\restriction(\gamma+1)\downarrow\nu)^\frown\vec{0} \in P_\nu$. That is, if $\gamma' \leq \gamma\downarrow\nu$ and $\nu_{\gamma'} > 0$ then there is $\gamma'' \in I^{\vec{\nu}^q} \cap \gamma+1$ such that $\gamma' \propto_{\nu^q_{\gamma''}} \gamma''$.

I think Proposition 1.12 was intended to help here. I'm not sure that it does, however. Also see the boxed comment after definition 1.10.

Probably have to remind reader what $q\|\bar{\gamma}$ is.

The proof of this claim is similar to that of lemma 4.9. Let $Q^*_{\bar{\mathcal{E}}}$ be the set of conditions $(\mathcal{E}, q) \in Q_\nu$ such that $\mathcal{E}\restriction\bar{\gamma}+1 = \bar{\mathcal{E}}$ and $q\|\bar{\gamma}+1 \in Q^{\bar{\mathcal{E}}}_\nu$. Then $G_\nu\|\bar{\gamma}+1$ is a generic subset of $Q^*_{\bar{\mathcal{E}}}$, so we can write $Q^*_{\bar{\mathcal{E}}} = Q^{\bar{\mathcal{E}}}_\nu * \dot{S}$ where S is the set of conditions $(\mathcal{E}, q) \in Q^*_{\bar{\mathcal{E}}}$ which are compatible with $G_\nu\|\bar{\gamma}+1$. As om the proof of Lemma 4.9, S is countable in $V[H_\nu, K_{\nu^++1}, G_\nu\|\bar{\gamma}+1]$. It follows that for any two conditions $(\mathcal{E}, q)$ and $(\mathcal{E}', q')$ in S there is an isomorphism $\pi\colon S/(\mathcal{E}, q) \cong S/(\mathcal{E}', q')$ in $V[H_\nu, K_{\nu^++1}, G_\nu\|\bar{\gamma}+1]$. If $(\mathcal{E}, q) \in G_\nu$ then this isomorphism induces a generic set $K' \cap \mathcal{R}^*_{\kappa+1}$ which is identical to K on $\mathcal{R}_\nu$ and $\mathcal{R}^*_{\nu+1,\kappa+1}$ and has $G'_\nu = \pi``G_\nu$. Then $V[H', K] = V[H, K]$ and all of the terms denoting parameters in the formula (13) have the same values when defined from H', K and when defined from H, K. It follows that $\dot{x}^{H',K} = \dot{x}^{H,K}$, and hence $(\mathcal{E}, q) \Vdash_{\mathcal{R}_\nu} \nu \in \dot{x}$ if and only if $(\mathcal{E}', q') \Vdash_{\mathcal{R}_\nu} \nu \in \dot{x}$, and similarly for $\nu \notin \dot{x}$.

Now $\mathcal{R}^{\bar{\mathcal{E}}}_{\nu+1} \equiv \mathcal{R}_\nu * \dot{P}^{\bar{\mathcal{E}}}_\nu$ since η is a successor ordinal, so the statement $\nu \in \dot{x}$ will be decided by some condition $(p, \xi) \in \mathcal{R}_\nu * P^{\bar{\mathcal{E}}}_\nu \cap H_{\nu+1}$. It follows that there are sets $X^+, X^- \subset \mathcal{R}_\kappa \times \nu$ so that if we let E be the set of ordinals $\nu \in E_\gamma$ such that

$$\left\{ (p,\xi) : (p,\xi) \Vdash_{\mathcal{R}_\nu * \dot{P}^{\bar{\mathcal{E}}}_\nu} \nu \in \dot{x} \right\} = X^+ \cap (\mathcal{R}_\nu \times \nu)$$
$$\left\{ (p,\xi) : (p,\xi) \Vdash_{\mathcal{R}_\nu * \dot{P}^{\bar{\mathcal{E}}}_\nu} \nu \notin \dot{x} \right\} = X^- \cap (\mathcal{R}_\nu \times \nu)$$

then E is a member of the repeat point W.

Now suppose that $\mathcal{E}_\kappa\restriction\eta+1 = \mathcal{E}^\frown\langle\eta, (E, h)\rangle$, and fix any condition $(p, \xi) \in X^+ \cup X^-$ so that $p \in H_\kappa$ and $\xi \in C_{\kappa,\eta-1}$. Let $\lambda = \max\{\sup(\operatorname{domain} p), \xi, \lambda_0\}+1$. Then $p^\frown\langle\kappa, (\eta+1, \xi)\rangle$ forces either $\dot{x} \setminus \lambda \subseteq C_{\kappa,\eta}$ or $\dot{x} \setminus \lambda \subseteq \kappa \setminus C_{\kappa,\eta}$, depending on whether $(p, \xi) \in X^+$ or $(p, \xi) \in X^-$. □

We showed in corollary 4.12 that the cardinals κ and κ^+ are preserved in N, in lemma 4.13 that N satisfies dependent choice, and in lemma 4.18 that the closed unbounded filter is an ultrafilter on $\kappa = \omega_1$ in N. This completes the proof of theorem 4.8 and hence of the theorem 1.1.

5 Questions

Question. What is the consistency strength of the assumption that the filter of closed unbounded subsets of ω_2 forms an ultrafilter, when restricted either to the ordinals of cofinality ω_1 or to those of cofinality ω?

This is not known to require any more large cardinal strength that the single repeat point needed in theorem 1.1 for ω_1. Note that lemma 4.9, and therefore lemmas 4.11 and 4.13, can be proved using the Levy collapse onto ω_1 rather than onto ω. Thus only the proof of lemma 4.18, asserting that the filter of closed, unbounded sets is an ultrafilter, relies on the use of the Levy collapse to make $\kappa = \omega_1$.

Even for larger cardinals, nothing more is know about the large cardinal strength needed to obtain a model in which the closed unbounded ultrafilter on κ^+ is an ultrafilter when restricted to ordinals of cofinality κ. In the case of the restriction to cardinals of cofinality $\lambda < \kappa$ covering lemma considerations, as in [6], come into play; however the best lower boundes which these methods seem to readily give are much lower than a strong cardinal, especially in the case $\lambda = \omega$.

In another direction, it would be of interest to know whether it is possible to vary the forcing of this paper so that κ and κ^+ are preserved in the generic extension M, which satisfies the axiom of choice.

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